CLASSIFICATION AND REPRESENTATION OF SEMI-SIMPLE JORDAN ALGEBRAS

BY

F. D. JACOBSON AND N. JACOBSON

In the present paper we use the term *special Jordan algebra* to denote a (non-associative) algebra $\mathfrak{R}$ over a field of characteristic not two for which there exists a 1–1 correspondence $a \rightarrow a^R$ of $\mathfrak{R}$ into an associative algebra $\mathfrak{A}$ such that

$$(1) \qquad (a + b)^R = a^R + b^R, \qquad (\alpha a)^R = \alpha a^R$$

for α in the underlying field and

$$(2) \qquad (a \cdot b)^R = (a^R b^R + b^R a^R)/2.$$

In the last equation the $\cdot$ denotes the product defined in the algebra $\mathfrak{R}$. When there is no risk of confusion we shall also use the $\cdot$ to denote the *Jordan product* $(xy+yx)/2$ in an associative algebra. Jordan multiplication is in general non-associative but it is easy to verify that the following special rules hold:

$$(3) \qquad a \cdot b = b \cdot a, \qquad (a \cdot b) \cdot a^2 = a \cdot (b \cdot a^2).$$

Hence these rules hold for the product in a special Jordan algebra.

Because of this fact one defines an abstract *Jordan algebra* to be a (non-associative) algebra in which the product satisfies (3). Such algebras were first studied by Jordan, Wigner and von Neumann(1), and recently Albert(2) has developed a successful structure theory for Jordan algebras over any field of characteristic 0. Using definitions of solvability and the radical that are customary for Lie algebras, he succeeded in carrying over to the Jordan case the known theorems of Lie, Engel and Cartan on solvable Lie algebras. Also he proved that an algebra that is semi-simple in the sense that it has no solvable ideals is a direct sum of simple algebras. The determination of simple Jordan algebras can be reduced to that of central simple algebras and for these Albert proved the existence of a finite extension field P of the base field such that $\mathfrak{R}_P$ is one of the following *split algebras*:

A. The algebra P_{nj} of $n \times n$ matrices over P relative to Jordan multiplication $a \cdot b = (ab+ba)/2$.

B. The subalgebra of P_{nj} of symmetric matrices.

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(1) Jordan, Wigner and von Neumann [1]. Numbers in brackets refer to the bibliography at the end of the paper.

(2) Albert [4]. Cf. also Albert [2].

C. The subalgebra of P_{nj}, $n=2m$, of matrices that are symmetric relative to the involution $a \rightarrow q^{-1}a'q$ where a' denotes the transposed of a and

$$q = \begin{pmatrix} 0 & 1_m \\ -1_m & 0 \end{pmatrix}. \tag{4}$$

D. The algebra Γ with basis $s_0, s_1, \cdots, s_n$ and multiplication table

$$\begin{aligned} s_0 \cdot s_i &= s_i, \\ s_i^2 = s_0, \qquad s_i \cdot s_j &= 0, \qquad i \neq j. \end{aligned} \tag{5}$$

E. The algebra of three rowed hermitian matrices with Cayley number coefficients relative to the composition $a \cdot b = (ab+ba)/2$. This algebra has dimensionality 27.

A Jordan algebra $\mathfrak{K}$ is said to be of *type* A, B, C, D or E if there exists a finite extension P of its base field Φ such that $\mathfrak{K}_P$ is one of the algebras in the list A, B, C, D or E respectively. As has been shown by Albert(3) any algebra of type D has a basis $s_0, s_1, \cdots, s_n$ such that

$$\begin{aligned} s_0 \cdot s_i &= s_i, \\ s_i^2 = \alpha_i s_0, \qquad s_i \cdot s_j &= 0, \qquad i \neq j = 1, 2, \cdots, n. \end{aligned}$$

The algebras of type B and C over a field of characteristic 0 have been determined by Kalisch(4). In the present paper we determine the algebras of type A. Our method is applicable for base fields of characteristics $\neq 2$ and we use it to show also that Kalisch's determination of the algebras of types B and C is valid with merely this restriction on the characteristic. We note finally that a determination of the algebras of type E has been given in a recent paper by R. Schafer(5).

A glance at the above list shows that the split algebras A, B, C and D are special Jordan algebras. It is not surprising that a similar result holds for the algebras of the corresponding types A–D. On the other hand it is known that the algebra E is not a special Jordan algebra(6) and at the present time a characterization of the special algebras is still lacking.

The present paper is not primarily concerned with the structure theory, or with the problem of characterizing the special Jordan algebras. The problem that we consider here is that of determining all the imbeddings of a special Jordan algebra in associative algebras. We define such an *imbedding* to be a homomorphism of $\mathfrak{K}$ into the Jordan algebra obtained by replacing ordinary multiplication in an associative algebra by Jordan multiplication. By definition, any special Jordan algebra possesses at least one isomorphic

(3) Albert [2, p. 548].
(4) Kalisch [1].
(5) Schafer [1].
(6) Albert [1].

(that is, 1–1) imbedding. Of particular interest are the imbeddings in matrix algebras, or what amounts to the same thing, in algebras of linear transformations. An imbedding of this type is called a *representation*. For these we have a natural concept of equivalence. Two representations R_1 and R_2 in Φ_m are called *equivalent* if there exists a matrix s in Φ_m such that

$$a^{R_2} = s^{-1}a^{R_1}s$$

for all a in $\mathfrak{K}$. We can also define *reducibility*, *decomposability* and *complete reducibility* of a representation in Φ_m in the usual fashion to mean reducibility, and so on, of the set $\mathfrak{K}^R$ of representing matrices a^R.

Of fundamental importance in the study of an imbedding R of $\mathfrak{K}$ in $\mathfrak{A}$ is the *enveloping algebra* $\mathfrak{E}(\mathfrak{K}, R)$ defined to be the (associative) subalgebra of $\mathfrak{A}$ generated by the representing elements a^R. As was first observed by Birkhoff and Whitman, any Jordan algebra $\mathfrak{K}$ possesses a universal imbedding R_0 and a *universal* (enveloping) *associative algebra* $\mathfrak{U}=\mathfrak{E}(\mathfrak{K}, R_0)$ that has the following property: If R is any imbedding of $\mathfrak{K}$ then the correspondence $a^{R_0}\rightarrow a^R$ can be extended to a homomorphism of the associative algebra $\mathfrak{U}$ on the associative algebra $\mathfrak{E}(\mathfrak{K}, R)$(7). Clearly $\mathfrak{U}$ is unique in the sense of isomorphism. Hence we speak of *the* universal associative algebra of $\mathfrak{K}$.

It is clear that the determination of the universal associative algebra $\mathfrak{U}$ of $\mathfrak{K}$ reduces the study of the imbeddings of $\mathfrak{K}$ to that of the homomorphic mappings of the associative algebra $\mathfrak{U}$. We therefore consider the problem of finding the algebra $\mathfrak{U}$. We remark that for the case of the split algebra of type D the solution of this problem is well known and is given by the definition of the algebra of Clifford numbers.

In the present paper we obtain the universal associative algebras for all the special semi-simple Jordan algebras. An outline of the procedure is the following: We first show that if $\mathfrak{K}$ has an identity and is a direct sum of two algebras $\mathfrak{K}_1$ and $\mathfrak{K}_2$ then $\mathfrak{U}$ is a direct sum of the universal algebras of the $\mathfrak{K}_i$. Next we determine the universal algebras for the split Jordan algebras A, B, C and D and we use this determination to obtain the Jordan algebras of types A, B and C(8). We then determine the universal algebras of the Jordan algebras of types A–D. As applications we obtain the isomorphisms and the

(7) This result was announced at the Algebra conference at the University of Chicago, June, 1946. Birkhoff and Whitman also announced at this conference that they had determined the universal algebra of the split algebra A. (See Birkhoff and Whitman [1].) On the other hand, the present authors were in possession at this time of a result that amounted to a determination of the universal algebras of the split algebra B and had partial results on the algebras A and C. Stimulated by Birkhoff and Whitman's announced results we succeeded subsequently in completing the work presented here.

(8) It should be mentioned that Albert [2] has determined the representations for all the split algebras of characteristic 0 and, in fact, for the somewhat more general class of *reduced* algebras. His results, however, are not in a form that is suitable for our purposes. Moreover, the assumption of non-modularity of the base field is used in an essential fashion in his determination.

derivations of these algebras.

Our results reduce the problem of representation of semi-simple Jordan algebras to that of semi-simple associative algebras. For we show that the universal algebra of a semi-simple Jordan algebra of characteristic 0 is semi-simple and we determine the simple components. A corollary of this result is that every representation of a semi-simple Jordan algebra is completely reducible. Moreover, the irreducible representations can be obtained by using our results and the known theory of associative algebras.

1. **The universal associative algebra.** We have defined a special Jordan algebra to be an algebra $\mathfrak{K}$ for which there exists a 1–1 mapping $a \rightarrow a^R$ into an associative algebra $\mathfrak{A}$ such that (1) and (2) hold. Another way of stating this is the following: Let $\mathfrak{A}$ be any associative algebra. We define in $\mathfrak{A}$ the Jordan product

$$a \cdot b = (ab + ba)/2$$

and we consider the set $\mathfrak{A}$ relative to the addition and scalar operations defined in the algebra $\mathfrak{A}$ and relative to Jordan multiplication as multiplication. Since Jordan multiplication is distributive and homogeneous we obtain in this way a (non-associative) algebra. We denote this algebra as $\mathfrak{A}_j$ and call it *the Jordan algebra determined by* $\mathfrak{A}$.

We can now define a special Jordan algebra to be any algebra that is isomorphic to a subalgebra of some $\mathfrak{A}_j$. It is clear that this definition is equivalent to our previous one.

Now let $\mathfrak{K}$ be a special Jordan algebra and let R be any imbedding (not necessarily 1–1) of $\mathfrak{K}$ in an associative algebra $\mathfrak{A}$. Let $x_1, x_2, \cdots$ be a basis for $\mathfrak{K}$ over Φ and let

$$x_i \cdot x_j = \sum \gamma_{ijk} x_k, \tag{6}$$

γ's in Φ, be the multiplication table. Then in $\mathfrak{A}$ we have the relations

$$\overset{R}{x_i} \cdot \overset{R}{x_j} = \sum \gamma_{ijk} \overset{R}{x_k}$$

so that

$$\overset{R}{x_i} \overset{R}{x_j} = - \overset{R}{x_j} \overset{R}{x_i} + 2 \sum \gamma_{ijk} \overset{R}{x_k}. \tag{7}$$

Since the x_i form a basis for $\mathfrak{K}$ any representing element a^R is a linear combination of the x_i^R. Hence any element of the enveloping algebra $\mathfrak{E}(\mathfrak{K}, R)$ is a linear combination of monomials $x_{i_1}^R x_{i_2}^R \cdots x_{i_n}^R$. Now if $i_j \geqq i_{j+1}$ we can replace $x_{i_j}^R x_{i_{j+1}}^R$ in this product by

$$- \overset{R}{x}_{i_{j+1}} \overset{R}{x}_{i_j} + 2 \sum \gamma_{i_j i_{j+1} k} \overset{R}{x_k}.$$

A succession of such substitutions will yield an expression for $x_{i_1}^R x_{i_2}^R \cdots x_{i_n}^R$ as a linear combination of monomials of the form

$$(8) \qquad (x_1^R)^{\epsilon_1}(x_2^R)^{\epsilon_2} \cdots (x_s^R)^{\epsilon_s}$$

where the $\epsilon_i = 0, 1$ and $(\epsilon_1, \epsilon_2, \cdots, \epsilon_s) \neq (0, 0, \cdots, 0)$. Thus we see that if the dimensionality $(\mathfrak{K}:\Phi) = n$ then $(\mathfrak{E}:\Phi) \leqq 2^n - 1$.

We consider now the free algebra $\mathfrak{F}$ with basis $X_1, X_2, \cdots$ in 1–1 correspondence $X_i \rightarrow x_i$ with the basis for $\mathfrak{K}$. Let $\mathfrak{B}$ be the two-sided ideal in $\mathfrak{F}$ generated by the elements

$$(9) \qquad (X_iX_j + X_jX_i)/2 - \sum \gamma_{ijk}X_k$$

and let $\mathfrak{U}$ be the difference algebra $\mathfrak{F}/\mathfrak{B}$. If $\bar{x}_i$ denotes the residue class of X_i mod $\mathfrak{B}$ then

$$\bar{x}_i \cdot \bar{x}_j = \sum \gamma_{ijk}\bar{x}_k.$$

Hence the correspondence $a = \sum \alpha_i x_i \rightarrow \bar{a} = \sum \alpha_i \bar{x}_i$ is an imbedding of $\mathfrak{K}$ in $\mathfrak{U}$.

Definition. An imbedding $a \rightarrow \bar{a}$ of a special Jordan algebra $\mathfrak{K}$ is called a *universal imbedding* and its enveloping algebra $\mathfrak{U}$ is a *universal associative algebra* for $\mathfrak{K}$ if the correspondence $\bar{a} \rightarrow a^R$ determined by an arbitrary imbedding R can be extended to a homomorphism of the enveloping algebra $\mathfrak{U}$ onto the enveloping algebra $\mathfrak{E} = \mathfrak{E}(\mathfrak{K}, R)$.

We shall now show that the imbedding $a \rightarrow \bar{a}$ that we have constructed is universal. Since $\mathfrak{F}$ is a free algebra there is one and only one homomorphism of $\mathfrak{F}$ into $\mathfrak{E}$ sending the generator X_i into the generator x_i^R of $\mathfrak{E}$. This homomorphism maps the element (9) into

$$(x_i^R x_j^R + x_j^R x_i^R) - \sum \gamma_{ijk} x_k^R = 0.$$

Hence the kernel $\mathfrak{N}$ of the homomorphism includes the ideal $\mathfrak{B}$. It follows that this homomorphism induces a homomorphism of $\mathfrak{U} = \mathfrak{F}/\mathfrak{B}$ into $\mathfrak{E}$ mapping $\bar{x}_i$ into x_i^R. Evidently this is an extension of the mapping $\bar{a} \rightarrow a^R$ as required[9].

We note next that the mapping $a \rightarrow \bar{a}$ is 1–1. For we have assumed that there exists an isomorphic imbedding R. For such an R the x_i^R are linearly independent. Hence the $\bar{x}_i$ are also linearly independent and $a \rightarrow \bar{a}$ is 1–1.

Suppose now that $a \rightarrow a'$ is any universal imbedding and that $\mathfrak{U}'$ is its enveloping algebra. Then by definition $a' \rightarrow \bar{a}$ and $\bar{a} \rightarrow a'$ can be extended to homomorphisms between the enveloping algebras. These extensions are unique and it is clear that each is an isomorphism. In this sense any two universal imbeddings are equivalent. We shall therefore speak of *the* universal imbedding and *the* universal algebra. We shall also identify $\mathfrak{K}$ with the subset of $\mathfrak{U}$ representing it. We can do this since the universal imbedding is 1–1. Also we shall write a in place of $\bar{a}$. Hence the universal imbedding is the identity mapping $a \rightarrow a$.

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(9) These results on the existence and finiteness of the universal algebra were first discovered by Birkhoff and Whitman. See footnote 6.

We shall now show that the universal algebra $\mathfrak{U}=\mathfrak{F}/\mathfrak{B}$ possesses an involution relative to which the elements of $\mathfrak{K}$ are symmetric. First it is clear that the mapping

$$\sum \alpha_{i_1\cdots i_r} X_{i_1} \cdots X_{i_r} \to \sum \alpha_{i_1\cdots i_r} X_{i_r} \cdots X_{i_1}$$

is an involution in the free algebra $\mathfrak{F}$. The elements X_i and therefore the elements $X_iX_j+X_jX_i$ are symmetric relative to this involution. Hence the basis (9) of $\mathfrak{B}$ consists of symmetric elements. This implies that $\mathfrak{B}$ is invariant under the involution. Hence we have an induced involution J in $\mathfrak{U}=\mathfrak{F}/\mathfrak{B}$ defined by

$$\sum \alpha_{i_1\cdots i_r} x_{i_1} \cdots x_{i_r} \to \sum \alpha_{i_1\cdots i_r} x_{i_r} \cdots x_{i_1}.$$

Evidently the x_i and hence every $a\in\mathfrak{K}$ is J-symmetric. We note also that J is uniquely determined. Thus let K be any involution in $\mathfrak{U}$ leaving the elements of $\mathfrak{K}$ fixed. Then

$$\left(\sum \alpha_{i_1\cdots i_r} x_{i_1} \cdots x_{i_r}\right)^K = \sum \alpha_{i_1\cdots i_r} x_{i_r} \cdots x_{i_1}.$$

Since any element of $\mathfrak{U}$ has the form $\sum\alpha_{i_1\cdots i_r} x_{i_1} \cdots x_{i_r}$ this shows that $K=J$. We shall refer to J as the *fundamental involution* in $\mathfrak{U}$.

2. **Universal algebra of a direct sum.** We suppose now that $\mathfrak{U}$ is any associative algebra and that $\mathfrak{K}$ is a subalgebra of the special Jordan algebra $\mathfrak{U}_j$ determined by $\mathfrak{U}$. We assume moreover that the smallest (associative) subalgebra of $\mathfrak{U}$ containing $\mathfrak{K}$ is $\mathfrak{U}$ itself. This is a more general situation than that considered in the preceding section in which $\mathfrak{U}$ is the universal algebra of $\mathfrak{K}$.

If $a\in\mathfrak{K}$, $a\cdot a=(a^2+a^2)/2=a^2$ and by induction we see that the rth Jordan power $a^{\cdot r}=(a^{\cdot r-1})\cdot a=a^r$. Thus the powers of a single element of $\mathfrak{K}$ generate an associative subalgebra of $\mathfrak{U}$.

If e is an idempotent element in the special Jordan algebra $\mathfrak{K}$ then e is idempotent in the associative algebra $\mathfrak{U}$. Suppose now that a is an element of $\mathfrak{K}$ such that $e\cdot a=0$. Then $ea=-ae$ and $e^2a=-eae=ae^2$ so that $ea=ae$. It follows that $ea=ae=0$. In particular if e and f are idempotent elements of $\mathfrak{K}$ that are *orthogonal* in the sense that $e\cdot f=0$ then these elements are idempotent and orthogonal in $\mathfrak{U}$ in the usual sense that $ef=0=fe$.

Next let e be an idempotent element of $\mathfrak{K}$ and let a be an element of $\mathfrak{K}$ that has e as identity. Then $e\cdot a=a$ so that $a=(ea+ae)/2$. Hence

$$a = (ea + ae)/2 = (ea + ae)/4 + eae/2.$$

This implies that $(ea+ae)/2=eae$ and that $a=eae$. Hence $ea=a=ae$. Thus e is an identity for a in the associative algebra. In particular we see that if $\mathfrak{K}$ has an identity e then $ea=a=ae$ for every $a\in\mathfrak{K}$. Since $\mathfrak{K}$ generates $\mathfrak{U}$ this means that e is an identity for $\mathfrak{U}$.

We assume next that $\mathfrak{K}$ has an identity e and that $\mathfrak{K}$ is a direct sum

$\mathfrak{K}_1 \oplus \mathfrak{K}_2$ of the ideals $\mathfrak{K}_i$. Then if $a_i \in \mathfrak{K}_i$, $a_1 \cdot a_2 \in \mathfrak{K}_1 \cap \mathfrak{K}_2$ and hence $a_1 \cdot a_2 = 0$. Write $e = e_1 + e_2$ where $e_i \in \mathfrak{K}_i$. Then $a = e \cdot a = e_1 \cdot a + e_2 \cdot a = a_1 + a_2$ where $a_i = e_i \cdot a$ is in k_i. This implies that the e_i are idempotent and that e_i acts as an identity for $\mathfrak{K}_i$. As we have seen this implies that e_i is an identity in the associative algebra $\mathfrak{U}_i$ generated by $\mathfrak{K}_i$. Since e_1 and e_2 are orthogonal, $\mathfrak{U}_1\mathfrak{U}_2 = 0 = \mathfrak{U}_2\mathfrak{U}_1$. Since $\mathfrak{K} = \mathfrak{K}_1 + \mathfrak{K}_2$, $\mathfrak{U} = \mathfrak{U}_1 + \mathfrak{U}_2$. Thus $\mathfrak{U} = \mathfrak{U}_1 \oplus \mathfrak{U}_2$. This proves the following theorem.

THEOREM 1. *Let $\mathfrak{U}$ be an associative algebra and let $\mathfrak{K}$ be a (Jordan) subalgebra of $\mathfrak{U}_j$ such that $\mathfrak{K}$ generates $\mathfrak{U}$. Then if $\mathfrak{K}$ has an identity and is a direct sum $\mathfrak{K}_1 \oplus \mathfrak{K}_2$, $\mathfrak{U}$ is a direct sum $\mathfrak{U}_1 \oplus \mathfrak{U}_2$ where $\mathfrak{U}_i$ is the algebra generated by $\mathfrak{K}_i$.*

We suppose now that $\mathfrak{U}$ is the universal algebra of $\mathfrak{K}$. We wish to show that under the hypothesis of Theorem 1, $\mathfrak{U}_i$ is the universal algebra for $\mathfrak{K}_i$. Let $a_1 \rightarrow a_1^R$ be an imbedding of $\mathfrak{K}_1$ in the associative algebra $\mathfrak{A}_1$. We form the direct sum $\mathfrak{A}_1 \oplus \mathfrak{U}_2$ and consider the correspondence $a_1 + a_2 \rightarrow a_1^R + a_2$. This is an imbedding of $\mathfrak{K}$ in $\mathfrak{A}_1 \oplus \mathfrak{U}_2$. If $\mathfrak{E}_1$ is the enveloping algebra of R then the enveloping algebra of the representation $a_1 + a_2 \rightarrow a_1^R + a_2$ is $\mathfrak{E}_1 \oplus \mathfrak{U}_2$. Since $\mathfrak{U}$ is the universal algebra of $\mathfrak{K}$ the imbedding of $\mathfrak{K}$ can be extended to a homomorphism of $\mathfrak{U}$. The contraction of the latter homomorphism to $\mathfrak{U}_1$ is an extension of R. Since R is arbitrary this proves that $\mathfrak{U}_1$ is the universal algebra of $\mathfrak{K}_1$. A similar statement holds for $\mathfrak{K}_2$.

THEOREM 2. *Let $\mathfrak{K}$ be a special Jordan algebra with an identity and let $\mathfrak{U}$ be its universal algebra. Then if $\mathfrak{K} = \mathfrak{K}_1 \oplus \mathfrak{K}_2$, $\mathfrak{U} = \mathfrak{U}_1 \oplus \mathfrak{U}_2$ where $\mathfrak{U}_i$ is the algebra generated by $\mathfrak{K}_i$, and $\mathfrak{U}_i$ is the universal algebra of $\mathfrak{K}_i$.*

3. **Universal algebras of split Jordan algebras.** In this section we shall determine the universal algebras for certain algebras that appear to play the same role in the theory of special Jordan algebras that is played by the full matrix algebra in the associative theory. These are the special split algebras defined in the introduction.

Class A[10]. These are the special Jordan algebras Φ_{nj} where Φ_n is the full matrix algebra. If $\mathfrak{K} = \Phi_{nj}$, $\mathfrak{K}$ has a basis e_{ij}, $i, j = 1, 2, \cdots, n$, with multiplication table

$$2e_{ij} \cdot e_{kl} = \delta_{jk}e_{il} + \delta_{il}e_{kj}. \tag{10}$$

In particular the elements e_{ii} are orthogonal idempotent elements in $\mathfrak{K}$.

If $\mathfrak{U}$ is the universal associative algebra of $\mathfrak{K}$ then the elements e_{ii} are orthogonal idempotent elements of $\mathfrak{U}$. We now define

$$g_{ij} = e_{ii}e_{ij}e_{jj}, \qquad i \neq j. \tag{11}$$

[10] The determination of these universal algebras was made first by Birkhoff and Whitman. (See footnote 6.) Since these results are needed in the consideration of the split algebras C we derive them anew here.

Then

$$g_{ij} = e_{ii}e_{ij}e_{jj} = (e_{ij} - e_{ij}e_{ii})e_{jj} = e_{ij}e_{jj}.$$

Similarly $g_{ij}=e_{ii}e_{ij}$. We have the following relations

$$g_{ij}g_{kl} = e_{ij}e_{jj}e_{kk}e_{kl} = 0 \qquad \text{if } j \neq k, \tag{12}$$

and if i, j, k are not equal then

$$g_{ij}g_{jk} = e_{ii}e_{ij}e_{jk}e_{kk} = e_{ii}(e_{ik} - e_{jk}e_{ij})e_{kk} = e_{ii}e_{ik}e_{kk} - e_{ii}e_{jk}e_{ij}e_{kk}.$$

Since e_{ii} is idempotent and $e_{ii}\cdot e_{jk}=0$, $e_{ii}\, e_{jk}=0$. Hence

$$g_{ij}g_{jk} = e_{ii}e_{ik}e_{kk} = g_{ik}, \qquad i,\ j,\ k \text{ distinct.} \tag{13}$$

We now define

$$g_{ii} = g_{ij}g_{ji} \tag{14}$$

and we prove that this element is independent of j. Let i, j, k be distinct. Then by (13)

$$g_{ij}g_{ji} = g_{ij}(g_{jk}g_{ki}) = (g_{ij}g_{jk})g_{ki} = g_{ik}g_{ki}.$$

We wish to show that the elements g_{ij}, $i, j=1, 2, \cdots, n$, satisfy the multiplication table for matrix units. Since (12), (13) and (14) hold we need only to verify the table for products in which one of the factors is a g_{ii}.

Since $e_{ii}g_{ij}=g_{ij}=g_{ij}e_{jj}$,

$$g_{ii}g_{jk} = 0 = g_{kj}g_{ii}, \qquad i \neq j. \tag{15}$$

Next let $i\neq j$ and choose $k\neq i, j$. Then

$$g_{ii}g_{ij} = (g_{ik}g_{ki})g_{ij} = g_{ik}g_{kj} = g_{ij}. \tag{16}$$

Finally let i, j, k be distinct. Then

$$g_{ii}^2 = g_{ij}g_{ji}g_{ik}g_{ki} = g_{ij}g_{jk}g_{ki} = g_{ik}g_{ki} = g_{ii}. \tag{17}$$

In a similar manner we prove that $g_{jk}g_{ii}=\delta_{ki}g_{ji}$ and this proves our assertion.

We define next a second set of matrix units. We set

$$\begin{aligned} g_{n+i,n+j} &= e_{ii}e_{ji}e_{jj}, \qquad i \neq j, \\ g_{n+i,n+i} &= g_{n+i,n+j}g_{n+j,n+i}. \end{aligned} \tag{18}$$

Since the table (10) is unaltered under interchange of the two subscripts of each element it is clear that the elements $g_{n+i,n+j}$ multiply like matrix units too. As for the g_{ij} we have $g_{n+i,n+j}=e_{ii}e_{ji}=e_{ji}e_{jj}$, $i\neq j$.

We wish to prove now that the product of any g_{ij} by any $g_{n+k,n+l}$ is 0. We note first that since e_{kk} is idempotent and $e_{kk}\cdot e_{ij}=0$ for i, j, k distinct, then $e_{ij}e_{kk}=0=e_{kk}e_{ij}$. Hence

$$g_{ij}g_{n+j,n+k} = e_{ij}e_{jj}e_{kj}e_{kk} = e_{ij}e_{kj}e_{kk} = -e_{kj}e_{ij}e_{kk} = 0.$$

This implies that

$$g_{ij}g_{n+j,n+i} = g_{ij}g_{n+j,n+k}g_{n+k,n+i} = 0$$

and

$$g_{ij}g_{n+j,n+j} = g_{ij}g_{n+j,n+i}g_{n+i,n+j} = 0.$$

Since $g_{ii}e_{ii}=g_{ii}$ and $e_{jj}g_{n+j,n+j}=g_{n+j,n+j}$, $g_{ii}g_{n+j,n+j}=0$ if $i\neq j$. The element $g=\sum g_{ii}$ is an identity for the g_{ij} and $g'=\sum g_{n+j,n+j}$ is an identity for the $g_{n+i,n+j}$. Our relations show that

$$gg' = \sum g_{ii}g_{n+j,n+j} = 0.$$

It follows that $g_{ij}g_{n+k,n+l}=0$ for all i, j, k, l. Similarly we can prove that $g_{n+k,n+l}g_{ij}=0$.

Now let $\Phi_n^{(1)}$ denote the subspace of $\mathfrak{U}$ generated by the g_{ij} and $\Phi_n^{(2)}$ the subspace generated by the $g_{n+i,n+k}$. Then $\Phi_n^{(1)}$ is a subalgebra of $\mathfrak{U}$ and $\Phi_n^{(1)}\Phi_n^{(2)}=0=\Phi_n^{(2)}\Phi_n^{(1)}$. By (10)

$$\begin{aligned} e_{ij} &= e_{ii}e_{ij} + e_{ij}e_{ii} = g_{ij} + g_{n+j,n+i}, \qquad i \neq j,\\ e_{ii} &= e_{ii}(e_{ii}+e_{jj})e_{ii} = e_{ii}(e_{ij}e_{ji}+e_{ji}e_{ij})e_{ii}\\ &= g_{ij}g_{ji} + g_{n+i,n+j}g_{n+j,n+i}\\ &= g_{ii} + g_{n+i,n+i}. \end{aligned}$$

Thus all the $e_{ij}\in\Phi_n^{(1)}+\Phi_n^{(2)}$. Hence $\mathfrak{U}=\Phi_n^{(1)}\oplus\Phi_n^{(2)}$.

We have seen that each $\Phi_n^{(i)}$ is a homomorphic image of the $n\times n$ matrix algebra. Since the matrix algebra is simple, $\Phi_n^{(i)}=0$ or $\Phi_n^{(i)}$ is isomorphic to the $n\times n$ matrix algebra. We shall now show that the latter alternative holds for both values of i. For this purpose we consider the representation of $\mathfrak{R}$ defined by

$$e_{ij}^{R} = \left(\begin{array}{c|c} s_{ij} & 0 \\ \hline 0 & s_{ji} \end{array}\right)$$

where s_{ij} is an $n\times n$ matrix with 1 in the (i, j) position and 0's elsewhere. It is immediate that the e_{ij}^R satisfy (10). The representation R can be extended to a representation of $\mathfrak{U}$ that we shall also denote as R. We can verify that

$$(19)\qquad \begin{aligned} g_{ij}^{R} &= e_{ii}^{R}e_{ij}^{R}e_{jj}^{R} = \left(\begin{array}{c|c} s_{ij} & 0 \\ \hline 0 & 0 \end{array}\right), \qquad i\neq j,\\ g_{n+i,n+j}^{R} &= e_{ii}^{R}e_{ji}^{R}e_{jj}^{R} = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & s_{ij} \end{array}\right), \qquad i\neq j. \end{aligned}$$

From this it is immediate that the elements g_{ij}^R, $g_{n+i,n+j}^R$ are not equal to 0.

Hence the g_{ij} and $g_{n+i,n+j}$ are not equal to 0. It is also clear that the representation R of $\mathfrak{U}$ is 1–1. Thus R is also a universal imbedding. For the purpose of visualization it is preferable to use this representation. From now on we therefore take $\mathfrak{K}$ to be the set of matrices

$$\left(\begin{array}{c|c} a & 0 \\ \hline 0 & a' \end{array}\right), \tag{20}$$

a an $n\times n$ matrix with elements in Φ. The universal algebra $\mathfrak{U}$ is the set of matrices

$$\left(\begin{array}{c|c} a & 0 \\ \hline 0 & b \end{array}\right). \tag{21}$$

We note now that the mapping

$$\begin{pmatrix} a & \\ & b \end{pmatrix} \to \begin{pmatrix} b' & \\ & a' \end{pmatrix}$$

is an involution in $\mathfrak{U}$. Since the elements of $\mathfrak{K}$ are fixed relative to this involution we know that it coincides with the fundamental involution J in $\mathfrak{U}$. On the other hand it is clear from the above definition that $\mathfrak{K}$ is the complete set of J-symmetric elements of its universal algebra.

The case $n=2$ excluded here will be treated in our discussion of the algebras of class D.

Class B. An algebra of this class has a 1–1 representation as the set of $n\times n$ symmetric matrices over Φ. If we use this representation we see that $\mathfrak{K}$ has a basis

$$f_{ij} = f_{ji} = (e_{ij} + e_{ji})/2, \qquad i, j = 1, 2, \cdots, n,$$

where the e_{ij} are matrix units. Using the multiplication table for the e_{ij} we obtain

$$4f_{ij}\cdot f_{kl} = \delta_{jk}f_{il} + \delta_{jl}f_{ik} + \delta_{ik}f_{jl} + \delta_{il}f_{jk}. \tag{22}$$

This can be broken down to the following relations:

$$\begin{aligned} f_{ii}^{\cdot 2} &= f_{ii}. \\ 4f_{ij}^{\cdot 2} &= f_{ii} + f_{jj}, && i \neq j. \\ 2f_{ii}\cdot f_{ij} &= f_{ij}, && i \neq j. \\ 4f_{ij}\cdot f_{ik} &= f_{jk}, && i, j, k \text{ distinct}. \\ f_{ii}\cdot f_{jj} = f_{ii}\cdot f_{jk} &= f_{ij}\cdot f_{kl} = 0, && i, j, k, l, \text{ distinct}. \end{aligned} \tag{23}$$

In particular the elements f_{ii} are orthogonal idempotent elements of $\mathfrak{K}$.

Now let $\mathfrak{U}$ be the universal associative algebra of $\mathfrak{K}$. The elements f_{ii} are orthogonal idempotent elements of $\mathfrak{U}$. We now define

$$g_{ii} = f_{ii}, \qquad g_{ij} = 2f_{ii}f_{ij}f_{jj}, \qquad i \neq j, \tag{24}$$

and we shall show that these are matrix units. First we have

$$g_{ii}^2 = g_{ii}, \qquad g_{ii}g_{jj} = 0, \qquad i \neq j, \tag{25}$$

and

$$g_{ii}g_{jk} = \delta_{ij}g_{ik}, \qquad g_{jk}g_{ii} = \delta_{ik}g_{ji}. \tag{26}$$

It follows that

$$g_{ij}g_{kl} = 0, \qquad \text{if } j \neq k. \tag{27}$$

We note next that if $i \neq j$ then

$$g_{ij} = 2f_{ii}f_{ij}f_{jj} = 2(f_{ij} - f_{ij}f_{ii})f_{jj} = 2f_{ij}f_{jj}$$

and similarly $g_{ij} = 2f_{ii}f_{ij}$. Hence

$$g_{ij}g_{ji} = 4f_{ii}f_{ij}^2f_{ii} = f_{ii}(f_{ii} + f_{jj})f_{ii} = f_{ii} = g_{ii}. \tag{28}$$

Next let i, j, k be distinct. Then

$$\begin{aligned} g_{ij}g_{jk} &= 4f_{ii}f_{ij}f_{jk}f_{kk} \\ &= 2f_{ii}(f_{ik} - f_{jk}f_{ij})f_{kk} \\ &= 2f_{ii}f_{ik}f_{kk} \end{aligned}$$

since $f_{ij}f_{kk} = 0$. Thus

$$g_{ij}g_{jk} = g_{ik}. \tag{29}$$

This proves our assertion.

Let Φ_n denote the subspace of $\mathfrak{U}$ generated by the g_{ij}. Then Φ_n is a complete matrix subalgebra of $\mathfrak{U}$. Also we have

$$(g_{ij} + g_{ji})/2 = f_{ii}f_{ij} + f_{ij}f_{ii} = 2(f_{ii} \cdot f_{ij}) = f_{ij}.$$

Thus the $f_{ij} \in \Phi_n$ and $\mathfrak{U} = \Phi_n$.

Another way of stating this result is that we may take $\mathfrak{R}$ to be the set of $n \times n$ symmetric matrices and take the universal algebra $\mathfrak{U}$ to be the complete matrix algebra. It is immediate that the fundamental involution J in $\mathfrak{U}$ is the usual mapping $a \rightarrow a'$. Hence $\mathfrak{R}$ is the set of J-symmetric elements of $\mathfrak{U}$.

Class C. Any algebra of this class has a 1–1 representation as the subset of Φ_n, $n = 2m$, of matrices a such that $q^{-1}a'q = a$ where q is given by (4). We write

$$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

where the $a_{ij} \in \Phi_m$. Then the condition $q^{-1}a'q = a$ is equivalent to

$$a_{22} = a'_{11}, \qquad a'_{12} = -a_{12}, \qquad a'_{21} = -a_{21}. \tag{30}$$

Hence if the e_{kl} are matrix units then the algebra $\mathfrak{K}$ has the following basis:

$$\begin{aligned} h_{ij} &= e_{ij} + e_{m+j,m+i}, \\ f_{ij} &= -f_{ji} = e_{i,m+j} - e_{j,m+i}, \\ d_{ij} &= -d_{ji} = e_{m+i,j} - e_{m+j,i}, \qquad i, j = 1, 2, \cdots, m. \end{aligned} \tag{31}$$

The h_{ij} satisfy (10) and these elements generate a subalgebra of $\mathfrak{K}$ isomorphic to the Jordan algebra Φ_{mj} of class A. Also it is clear that the Jordan products of any two f's or of any two d's is 0. The following completes the multiplication table:

$$\begin{aligned} 2h_{ij}\cdot f_{kl} &= \delta_{jk}f_{il} + \delta_{jl}f_{ki}, \\ 2h_{ij}\cdot d_{kl} &= \delta_{il}d_{kj} + \delta_{ik}d_{jl}, \\ 2f_{ij}\cdot d_{kl} &= \delta_{jk}h_{il} + \delta_{il}h_{jk} - \delta_{ik}h_{jl} - \delta_{jl}h_{ik}. \end{aligned} \tag{32}$$

We shall assume that $m \geqq 3$ in the remainder of our discussion. We consider $\mathfrak{K}$ now as imbedded in its universal algebra $\mathfrak{U}$ and we wish to determine the structure of $\mathfrak{U}$. We define the following elements in $\mathfrak{U}$:

$$\begin{aligned} g_{ij} &= h_{ii}h_{ij}h_{jj}, & i \neq j, \\ g_{ii} &= g_{ij}g_{ji}, \\ g_{m+i,m+j} &= h_{ii}h_{ji}h_{jj}, & i \neq j, \\ g_{m+i,m+i} &= g_{m+i,m+j}g_{m+j,m+i}, \\ g_{i,m+j} &= h_{ii}f_{ij}h_{jj}, & i \neq j, \\ g_{i,m+i} &= g_{ij}g_{j,m+i}, \\ g_{m+i,j} &= h_{ii}d_{ij}h_{jj}, & i \neq j, \\ g_{m+i,i} &= g_{m+i,j}g_{ji}, \end{aligned} \tag{33}$$

$i, j = 1, 2, \cdots, m$. We shall show that these elements are uniquely defined and that they satisfy the multiplication rules for matrix units. This has already been established for the elements g_{ij} and $g_{m+i,m+j}$ in our discussion of class A. We recall also that $g_{ij} = h_{ii}h_{ij} = h_{ij}h_{jj}$ and $g_{m+i,m+j} = h_{ii}h_{ji} = h_{ji}h_{jj}$.

We note next that if $i \neq j$ then

$$g_{i,m+j} = h_{ii}f_{ij}h_{jj} = (f_{ij} - f_{ij}h_{ii})h_{jj} = f_{ij}h_{jj}$$

since the h_{ii} are orthogonal idempotent elements. Similarly $g_{i,m+j} = h_{ii}f_{ij}$ and $g_{m+i,j} = h_{ii}d_{ij} = d_{ij}h_{jj}$. Let i, j, k be distinct. Then

$$g_{kj}g_{j,m+i} = h_{kk}h_{kj}f_{ji}h_{ii} = h_{kk}(f_{ki} - f_{ji}h_{kj})h_{ii} = h_{kk}f_{ki}h_{ii} = g_{k,m+i}. \tag{34}$$

Hence

$$g_{ij}g_{j,m+i} = g_{ik}g_{kj}g_{j,m+i} = g_{ik}g_{k,m+i}.$$

Hence

$$g_{ij}g_{j,m+i} = g_{i,m+i} \tag{35}$$

for all j. Also

$$\begin{aligned} g_{jj}g_{j,m+i} &= g_{jk}g_{kj}g_{j,m+i} = g_{jk}g_{k,m+i} \\ &= g_{j,m+i}, \end{aligned} \tag{36}$$

and these relations imply that

$$g_{hk}g_{j,m+i} = g_{hk}g_{jj}g_{j,m+i} = \delta_{jk}g_{hj}g_{j,m+i} = \delta_{jk}g_{h,m+i} \tag{37}$$

for all h, k, i, j provided that $i \neq j$. But $g_{i,m+i} = g_{ij}g_{j,m+i}$. Hence

$$g_{hk}g_{i,m+i} = g_{hk}g_{ij}g_{j,m+i} = \delta_{ik}g_{hj}g_{j,m+i} = \delta_{ik}g_{h,m+i}. \tag{38}$$

We next prove that if $i \neq j$

$$\begin{aligned} g_{i,m+j}g_{m+j,m+i} &= h_{ii}f_{ij}h_{ij}h_{ii} = -h_{ii}h_{ij}f_{ij}h_{ii} \\ &= h_{ii}h_{ij}f_{ji}h_{ii} \\ &= g_{ij}g_{j,m+i} \\ &= g_{i,m+i}. \end{aligned}$$

Using this relation we can prove in the same manner that (37) and (38) were established that

$$g_{j,m+i}g_{m+h,m+k} = \delta_{ih}g_{j,m+k} \tag{39}$$

for all h, i, j, k.

We now note that

$$g_{j,m+i}g_{hk} = 0 = g_{m+h,m+k}g_{j,m+i}. \tag{40}$$

This is clear since $g_{j,m+i}g_{m+i,m+i} = g_{j,m+i} = g_{jj}g_{j,m+i}$ while $g_{m+i,m+i}g_{hk} = 0 = g_{m+h,m+k}g_{jj}$.

In a similar manner we can prove

$$g_{m+i,j}g_{hk} = \delta_{jh}g_{m+i,k}, \tag{41}$$

$$g_{m+h,m+k}g_{m+i,j} = \delta_{ik}g_{m+h,j}, \tag{42}$$

$$g_{hk}g_{m+i,j} = 0 = g_{m+i,j}g_{m+h,m+k}. \tag{43}$$

It remains to consider the products $g_{h,m+k}g_{m+i,j}$ and $g_{m+i,j}g_{h,m+k}$. We have

$$g_{h,m+k}g_{m+i,j} = g_{h,m+k}g_{m+k,m+k}g_{m+i,m+i}g_{m+i,j} = 0$$

if $i \neq k$. Let h, i, j be distinct. Then

$$g_{h,m+i}g_{m+i,j} = h_{hh}f_{hi}d_{ij}h_{jj} = h_{hh}(h_{hj} - d_{ij}f_{hi})h_{jj}$$
$$(44) \qquad = h_{hh}h_{hj}h_{jj}$$
$$= g_{hj}.$$

This implies that

$$g_{h,m+i}g_{m+i,h} = g_{h,m+i}g_{m+i,j}g_{jh} = g_{hj}g_{jh} = g_{hh}.$$

In a similar manner we can use the definition (33) to extend (44) to the cases $h=i\neq j$, $h\neq i=j$ and $h=i=j$. A like argument yields

$$(45) \qquad g_{m+i,j}g_{h,m+k} = \delta_{jh}g_{m+i,m+k}.$$

Thus we have proved that the g's are matrix units. We now show that the subalgebra Φ_n, $n=2m$, determined by these units coincides with $\mathfrak{U}$. This follows from the following equations:

$$g_{ij} + g_{j+m,i+m} = h_{ii}h_{ij} + h_{ij}h_{ii} = h_{ij}, \qquad i \neq j,$$
$$g_{ii} + g_{m+i,m+i} = g_{ij}\, g_{ji} + g_{m+i,m+j}g_{m+j,m+i}$$
$$= h_{ii}h_{ij}\, h_{ji}h_{ii} + h_{ii}h_{ji}h_{ij}\, h_{ii}$$
$$= h_{ii}(h_{ij}h_{ji} + h_{ji}h_{ij})h_{ii}$$
$$= h_{ii}(h_{ii} + h_{jj})h_{ii}$$
$$= h_{ii},$$
$$g_{i,m+j} - g_{j,m+i} = h_{ii}f_{ij} - f_{ji}h_{ii} = h_{ii}f_{ij} + f_{ij}h_{ii} = f_{ij},$$
$$g_{i+m,j} - g_{j+m,i} = h_{ii}d_{ij} - d_{ji}h_{ii} = h_{ii}d_{ij} + d_{ij}h_{ii} = d_{ij}.$$

For these equations show that the f_{ij}, d_{ij} and h_{ij} are in Φ_n. Hence $\mathfrak{U}=\Phi_n$.

This discussion shows that we can take $\mathfrak{K}$ to be the set of matrices satisfying (30) and $\mathfrak{U}$ to be the complete set of $n\times n$ matrices. The involution J in the universal algebra has the form $a\to q^{-1}a'q$, and $\mathfrak{K}$ is the set of J-symmetric elements of $\mathfrak{U}$.

Class D. We consider a Jordan algebra $\mathfrak{K}$ that has a basis of $n+1$ elements $s_0, s_1, \cdots, s_n$ and multiplication table

$$s_0\cdot s_i = s_i,$$
$$(46) \qquad s_i^2 = \alpha_i s_0, \qquad \alpha_i \neq 0,\ i = 1, 2, \cdots, n,$$
$$s_i\cdot s_j = 0, \qquad i, j = 1, \cdots, n;\ i \neq j.$$

The universal algebra $\mathfrak{U}$ of $\mathfrak{K}$ is the well known algebra of Clifford numbers[11]. It has the basis

$$(47) \qquad s_1^{\epsilon_1}s_2^{\epsilon_2}\cdots s_n^{\epsilon_n}$$

[11] See Weyl and Brauer [1], Witt [1].

where the $\epsilon_j = 0$, 1 and the multiplication table for this basis can be deduced by means of the associative law from the basic relations

(48) $$s_i^2 = \alpha_i, \qquad s_i s_j = -s_j s_i, \qquad i \neq j\text{[12]}.$$

In determining the structure of $\mathfrak{U}$ it is necessary to distinguish the two cases n even and n odd. We consider first the case $n = 2m$. Here we introduce the elements

(49) $$u_m = s_1 s_2 \cdots s_{2m-1}, \qquad v_m = s_{2m-1} s_{2m}$$

and we deduce from (48) that

(50) $$u_m^2 = (-1)^{m-1} \alpha_1 \alpha_2 \cdots \alpha_{2m-1}, \quad v_m^2 = -\alpha_{2m-1}\alpha_{2m}, \quad u_m v_m = -v_m u_m.$$

Thus the subalgebra Q_m that has the basis $(s_0, u_m, v_m, u_m v_m)$ is a generalized quaternion algebra. It follows that $\mathfrak{U} = Q_m \times \mathfrak{V}$ where $\mathfrak{V}$ is the subalgebra of $\mathfrak{U}$ of elements that commute with all the elements of Q_m. Since the dimensionality $(\mathfrak{U}:\Phi) = 2^{2m}$, $(\mathfrak{V}:\Phi) = 2^{2(m-1)}$. On the other hand it is easy to verify that the elements $s_1, s_2, \cdots, s_{2m-2}$ commute with u_m and v_m. These s_i generate a Clifford algebra of dimension $2^{2(m-1)}$. Hence this system coincides with $\mathfrak{V}$. An inductive argument now yields the formula

$$\mathfrak{U} = Q_1 \times Q_2 \times \cdots \times Q_m,$$

where $Q_k = (s_0, u_k, v_k, u_k v_k)$ and

(51) $$u_k^2 = (-1)^k \alpha_1 \alpha_2 \cdots \alpha_{2k-1}, \qquad v_k^2 = -\alpha_{2k-1}\alpha_{2k}.$$

It follows that $\mathfrak{U}$ is a central simple associative algebra.

We consider next the case of an odd n. Set $n = 2m+1$ and

$$c = s_1 s_2 \cdots s_{2m+1}.$$

Evidently c commutes with every s_i. Hence c is an element $\neq s_0$ in the center $\mathfrak{C}$ of $\mathfrak{U}$. We have

(52) $$c^2 = (-1)^m \alpha_1 \alpha_2 \cdots \alpha_{2m+1}.$$

Hence $\Phi(c)$ is either a quadratic field over Φ or is a direct sum of two algebras of order one. These alternatives hold according as $(-1)^m \prod_1^{2m+1} \alpha_i$ is not or is square in Φ. The elements $s_1, s_2 \cdots, s_{2m}$ generate a Clifford algebra $\mathfrak{W}$ which is central simple of dimensionality 2^{2m}. Since s_{2m+1} is a multiple of $cs_1s_2 \cdots s_{2m}$ the space $\mathfrak{W}\Phi(c) = \Phi(c)\mathfrak{W} = \mathfrak{U}$. Hence $\mathfrak{U} = \mathfrak{W} \times \Phi(c)$. Since $\mathfrak{W}$ is central this implies that $\Phi(c) = \mathfrak{C}$. Moreover, if we refer to the structure of $\Phi(c)$ we see that either $\mathfrak{U}$ is simple with $\Phi(c)$ as center or $\mathfrak{U}$ is a direct sum of two central simple algebras over Φ each isomorphic to the Clifford algebra $\mathfrak{W}$.

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[12] The defining relations (46) show that s_0 is the identity in U. Hence we suppress it in our formulas. Thus we write $s_i^2 = \alpha_i$ instead of $s_i^2 = \alpha_i s_0$.

Suppose now that all the $\alpha_i = 1$ and that Φ contains $(-1)^{1/2}$. Then (51) reads

$$u_k^2 = (-1)^k, \qquad v_k^2 = -1.$$

Hence Q_k is a complete matrix algebra of 2 rows and columns. If n is even then $\mathfrak{U}$ is a direct product of m such algebras. Hence $\mathfrak{U}$ is isomorphic to the complete matrix algebra of $2m$ rows. If $n = 2m+1$, $\Phi(c)$ is a direct sum of two algebras of order one and $\mathfrak{U}$ is a direct sum of two complete matrix algebras on $2m$ rows.

We shall now show that the two cases excluded before, namely, class A with $n = 2$ and class C with $m = 2$ can also be regarded as algebras of the present type. In the first case we choose as basis for $\mathfrak{K}$:

$$s_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad s_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then the s_i satisfy (46) with $\alpha_1 = 1$, $\alpha_2 = -1$, $\alpha_3 = 1$. Hence $c^2 = 1$ and $\mathfrak{U}$ is a direct sum of two algebras each of which is isomorphic to the Clifford algebra determined by s_1, s_2. Since $s_1^2 = s_0$, $s_1^2 = -s_0$ this algebra is isomorphic to the complete matrix algebra of two rows. This proves that the result obtained for algebras of class A is also valid for the case $n = 2$.

Now let $\mathfrak{K}$ be the algebra of class C with $m = 2$. We take the following basis in $\mathfrak{K}$:

$$s_0 = \left(\begin{array}{cc|cc} 1 & & & \\ & 1 & & \\ \hline & & 1 & \\ & & & 1 \end{array}\right), \quad s_1 = \left(\begin{array}{cc|cc} 1 & & & \\ & -1 & & \\ \hline & & 1 & \\ & & & -1 \end{array}\right),$$

$$s_2 = \left(\begin{array}{cc|cc} 0 & 1 & & \\ -1 & 0 & & \\ \hline & & 0 & -1 \\ & & 1 & 0 \end{array}\right), \quad s_3 = \left(\begin{array}{cc|cc} 0 & 1 & & \\ 1 & 0 & & \\ \hline & & 0 & 1 \\ & & 1 & 0 \end{array}\right),$$

$$s_4 = \left(\begin{array}{cc|cc} & & 0 & 1 \\ & & -1 & 0 \\ \hline 0 & 1 & & \\ -1 & 0 & & \end{array}\right), \quad s_5 = \left(\begin{array}{cc|cc} & & 0 & 1 \\ & & -1 & 0 \\ \hline 0 & -1 & & \\ 1 & 0 & & \end{array}\right).$$

It can be verified that (46) holds with $\alpha_1 = \alpha_3 = \alpha_5 = 1$, $\alpha_2 = \alpha_4 = -1$. Hence $c^2 = 1$ and $\mathfrak{U}$ is a direct sum of two algebras of the form $Q_1 \times Q_2$ where $Q_k = (s_0, u_k, v_k, u_k v_k)$ and

$$u_1^2 = -1, \quad v_1^2 = 1; \qquad u_2^2 = -1, \quad v_2^2 = 1.$$

Thus $Q_1 \times Q_2$ is a complete matrix algebra of four rows. We therefore see that the universal algebra is a direct sum of two matrix algebras and this result is different from that obtained for the other algebras of class C.

4. **Determination of the algebras of types A, B, C.** We say that a Jordan algebra $\mathfrak{K}$ is of *type* A, B, or C if there exists a finite extension P of the base field $\mathfrak{K}$ such that $\mathfrak{K}_P$ is an algebra of class A, B, or C respectively. In this section we determine these algebras. We shall assume that $m \geqq 3$ if $\mathfrak{K}$ has type C. We begin with the simpler

Types B *and* C. If $\mathfrak{K}$ is of type B we may regard $\mathfrak{K}$ as a Φ-subalgebra of the Jordan algebra of symmetric matrices over P such that the P space $P\mathfrak{K}$ spanned by these matrices is the complete set of symmetric matrices. Similarly if $\mathfrak{K}$ is of type C we may take $\mathfrak{K}$ to be a Φ-subalgebra of the Jordan algebra of matrices satisfying (30) such that $P\mathfrak{K}$ is the complete set of these matrices. In either case if $x_1, x_2, \cdots, x_r$ is a basis for $\mathfrak{K}$ then these x's also constitute a basis for the extended system. Since $\mathfrak{K}$ is a Φ-subalgebra

$$x_i \cdot x_j = \sum \gamma_{ijk} x_k, \tag{53}$$

where the γ's are in Φ and as usual $x_i \cdot x_j = (x_i x_j + x_j x_i)/2$.

Let $\rho \rightarrow \rho^R$ be a regular representation of the field P over Φ. The matrices ρ^R are in the matrix algebra Φ_h if $h = (P:\Phi)$. If $\alpha \in \Phi$ then α^R is the scalar matrix α. If $x = (\xi_{ij})$ is in P_n we define x^{R^*} to be the matrix in Φ_{nh} obtained by replacing each ξ_{ij} by the "block" $\xi_{ij}^R \in \Phi_h$. The correspondence $R^*: x \rightarrow x^{R^*}$ is a representation of P_n regarded as an algebra over Φ. Hence by (53)

$$x_i^{R^*} \cdot x_j^{R^*} = \sum \gamma_{ijk} x_k^{R^*}. \tag{54}$$

It follows that if the $\xi_i \in P$ then the correspondence $\sum \xi_i x_i \rightarrow \sum \xi_i x_i^{R^*}$ is a representation of $\mathfrak{K}_P$ over P. We know that this can be extended to a representation R' of the universal algebra P_n of $\mathfrak{K}_P$.

Now R^* and R' are both homomorphisms of $\mathfrak{P}_n$ into the ring $\mathfrak{P}_{nh}$ and if $\alpha \in \Phi$ then

$$(\alpha x)^{R^*} = \alpha x^{R^*}, \qquad (\alpha x)^{R'} = \alpha x^{R'}.$$

It follows that the totality $\mathfrak{A}$ of elements x such that $x^{R^*} = x^{R'}$ is a Φ-subalgebra of P_n. Since $x_i^{R^*} = x_i^{R'}$, $\mathfrak{K} \subseteq \mathfrak{A}$. Hence the Φ-subalgebra $\mathfrak{E}$ generated by $\mathfrak{K}$ is contained in $\mathfrak{A}$.

Since P_n is generated by $\mathfrak{K}_P$ and the x_i form a basis for $\mathfrak{K}_P$ we can adjoin to the basis $x_1, x_2, \cdots, x_r$ suitable products of these x's to obtain a basis $x_1, x_2, \cdots, x_{n^2}$ for P_n over P. These $x_j \in \mathfrak{E}$ and hence to $\mathfrak{A}$. Hence every $\sum_1^{n^2} \alpha_j x_j$, α_j in Φ is in $\mathfrak{A}$. Now let $x = \sum \xi_j x_j$ be any element of $\mathfrak{A}$. Then

$$\sum \xi_j^{R^*} x_j^{R^*} = \left(\sum \xi_j x_j\right)^{R^*} = \left(\sum \xi_j x_j\right)^{R'} = \sum \xi_j x_j^{R'} = \sum \xi_j x_j^{R^*}. \tag{55}$$

We wish to conclude from this relation that $\xi_j^{R^*}=\xi_j$. This can be done by using the following lemma.

LEMMA. *Let C be the subring of P_{nh} of matrices of the form $\sum \rho_j^{R^*}\sigma_j$ where the ρ's and σ's are in P. Then if $x_1, x_2, \cdots, x_{n^2}$ is a basis for P_n over P, the matrices $x_j^{R^*}$ are left linearly independent over C.*

Let $\{e_{ij}\}$ be a set of matrix units for P_n and set $E_{ij}=e_{ij}^{R^*}$. Then we have $E_{ij}E_{kl}=\delta_{jk}E_{il}$ and $\sum E_{ii}$ is the identity in P_{nh}. Since $\rho e_{ij}=e_{ij}\rho$ for any $\rho\in$P, $\rho^{R^*}E_{ij}=E_{ij}\rho^{R^*}$. Hence $cE_{ij}=E_{ij}c$ for any $c\in C$. Suppose that $\sum c_{ij}E_{ij}=0$, c_{ij} in C. Then

$$c_{pq}=\sum_k E_{kp}\left(\sum_{i,j} c_{ij}E_{ij}\right)E_{qk}=0.$$

We now write $e_1, e_2, \cdots, e_{n^2}$ for the e_{ij} and express $x_j=\sum \xi_{jk}e_k$. Reciprocally $e_k=\sum \eta_{kj}x_j$ and the matrices (ξ) and (η) are inverses. Now we have the relations $x_j^{R^*}=\sum \xi_{jk}^{R^*}e_k^{R}$ so that if $\sum c_jx_j^{R^*}=0$, then

$$\sum c_j\xi_{jk}^{R^*}e_k^{R^*}=0.$$

Since the elements $\sum c_j\xi_{jk}^{R^*}\in C$ this implies that

$$\sum_j c_j\xi_{jk}^{R^*}=0, \qquad k=1, 2, \cdots, n^2.$$

If we multiply by $\eta_{kl}^{R^*}$ and sum on k we obtain $c_i=0$ for all i. This proves the lemma.

We now see that if $\sum_1^{n^2}\xi_jx_j\in\mathfrak{A}$ then $\xi_j^{R^*}=\xi_j1$. Hence $\xi_j^{R}=\xi_j1$. It is known that if ρ is an element of P such that $\rho^R=\rho 1$ then $\rho\in\Phi$(13). Thus $\mathfrak{A}$ coincides with the totality of elements $\sum\alpha_jx_j$, α_j in Φ. Since $x_j\in\mathfrak{E}$ this shows that $\mathfrak{A}\subseteq\mathfrak{E}$. Hence $\mathfrak{E}=\mathfrak{A}$.

Since the x_j, $j=1, \cdots, n^2$, form a basis for $\mathfrak{A}$ and also a basis for P_n over P we see that $\mathfrak{A}_P=P_n$. Hence $\mathfrak{A}$ is a central simple associative algebra over Φ.

Since the involution J leaves the elements of $\mathfrak{R}$ fixed, J maps $\mathfrak{A}$ into itself. Hence J induces an involution in $\mathfrak{A}$. Let y be an element of $\mathfrak{A}$ that is J-symmetric. Then y is P-dependent on the basis $x_1, x_2, \cdots, x_r$ of $\mathfrak{R}_P$. It follows that y is Φ-dependent on those elements so that $y\in\mathfrak{R}$. Thus $\mathfrak{R}$ may be characterized as the totality of J-symmetric elements of $\mathfrak{A}$. This proves the following theorem.

THEOREM 3. *Let $\mathfrak{R}$ be a Jordan algebra of type* B *or* C *over the field Φ. Then $\mathfrak{R}$ is isomorphic to the subalgebra of J-symmetric elements of a Jordan algebra $\mathfrak{A}_j$ where $\mathfrak{A}$ is a central simple associative algebra that possesses an involution J*(14).

(13) N. Jacobson [2, p. 21].

(14) For algebras of characteristic 0 this result is due to Kalisch [1].

Type A. If $\mathfrak{R}$ is a Jordan algebra of type A then $\mathfrak{R}$ can be regarded as a Φ-subalgebra of the Jordan algebra of matrices of the form

$$\begin{pmatrix} a & \\ & a' \end{pmatrix}$$

and the P-space $\mathrm{P}\mathfrak{R}$ is the complete set of these matrices. The argument that we shall use to determine the structure of $\mathfrak{R}$ will parallel that given in the B and C cases.

If $x_1, x_2, \cdots, x_{n^2}$ is a basis for $\mathfrak{R}$ over Φ then these x's form a basis for $\mathfrak{R}_{\mathrm{P}}$ over P and we have a Jordan multiplication table of the form (53) with the γ's in Φ. Also we know that the universal algebra $\mathfrak{U}$ of $\mathfrak{R}_{\mathrm{P}}$ is the set of matrices

$$\begin{pmatrix} a & \\ & b \end{pmatrix}$$

where a and b are arbitrary in P_n. For any element $z=(\xi_{ij})$ of this algebra we define z^{R*} to be the matrix obtained by replacing the ξ_{ij} by the $h\times h$ matrices ξ_{ij}^R representing these elements in the regular representation R. We know that the elements x_i^{R*} satisfy (54). Hence the correspondence $\sum \xi_i x_i \rightarrow \sum \xi_i x_i^{R*}$, ξ_i in P, is a representation of $\mathfrak{R}_{\mathrm{P}}$ over P. We know that this representation can be extended to a representation of $\mathfrak{U}$ over P.

Let $\mathfrak{A}$ denote the totality of elements $x\in\mathfrak{U}$ such that $x^{R*}=x^{R'}$. Then $\mathfrak{A}$ is a Φ-subalgebra of $\mathfrak{U}$ containing $\mathfrak{R}$ and hence containing the Φ-subalgebra $\mathfrak{E}$ of $\mathfrak{U}$ generated by $\mathfrak{R}$. We can find a basis $x_1, \cdots, x_{n^2}, x_{n^2+1}, \cdots, x_{2n^2}$ for $\mathfrak{U}$ over P such that the x_j, $j>n^2$, are products of x_i, $i\leqq n^2$. Thus all the x's are in $\mathfrak{E}$. Now let $\sum_1^{2n^2}\xi_j x_j$ be any element of $\mathfrak{A}$. Then as in (55), $\sum \xi_i x_j^{R*} = \sum \xi_j^{R*} x_j^{R*}$.

The lemma given above now implies that $\xi_j^{R*}=\xi_j$ for all j.

We can now show as before that $\sum \xi_j x_j \in \mathfrak{A}$ if and only if the $\xi_j=\alpha_j$ are in Φ. This implies that $\mathfrak{A}=\mathfrak{E}$. Also we see that $\mathfrak{A}_{\mathrm{P}}=\mathfrak{U}$. If we use the fact that $\mathfrak{U}$ is a direct sum $\mathrm{P}_n^{(1)}\oplus \mathrm{P}_n^{(2)}$ of two complete matrix algebras we obtain the following possibilities for the structure of $\mathfrak{A}$: (1) $\mathfrak{A}$ is a direct sum of two central simple algebras $\mathfrak{A}_i$ over Φ, (2) $\mathfrak{A}$ is simple with center a quadratic extension $\Phi(q)$ isomorphic to a subfield of P. We consider these cases separately.

Case 1. $\mathfrak{A}=\mathfrak{A}_1\oplus\mathfrak{A}_2$. Then $\mathfrak{A}_{\mathrm{P}}=\mathfrak{A}_{1\mathrm{P}}\oplus\mathfrak{A}_{2\mathrm{P}}=\mathrm{P}_n^{(1)}\oplus\mathrm{P}_n^{(2)}$. Because of the uniqueness of the decomposition of an algebra as a direct sum of simple algebras we can suppose that $\mathfrak{A}_{i\mathrm{P}}=\mathrm{P}_n^{(i)}$. Since $\mathfrak{A}$ is the algebra generated by $\mathfrak{R}$, the fundamental involution J in $\mathfrak{U}$ induces an involution in $\mathfrak{A}$. Since J maps $\mathrm{P}_n^{(1)}$ on $\mathrm{P}_n^{(2)}$ and $\mathrm{P}_n^{(2)}$ on $\mathrm{P}_n^{(1)}$, J interchanges the two components $\mathfrak{A}_i$. Thus $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are anti-isomorphic. It is easy to see that $\mathfrak{R}$ is the set of J-symmetric elements of $\mathfrak{A}$. Hence $\mathfrak{R}$ is the totality of elements $a_1+a_1^J$, a_1 in $\mathfrak{A}_1$. The correspondence $a_1+a_1^J\rightarrow a_1$ is an isomorphism of $\mathfrak{R}$ on the Jordan algebra $\mathfrak{A}_{1j}$. Thus $\mathfrak{R}$ is isomorphic to a Jordan algebra $\mathfrak{A}_{1j}$ where $\mathfrak{A}_1$ is central simple over Φ.

Case 2. $\mathfrak{A}$ is simple with center a quadratic extension $\Phi(q)$ of Φ. Since $\mathfrak{A}_P$ is a direct sum,

$$\Phi(q)_P = P^{(1)} \oplus P^{(2)}$$

where each P_i is one-dimensional. Thus $P^{(i)} = Pe_i$ where

$$e_1 + e_2 = 1, \qquad e_1e_2 = e_2e_1 = 0, \qquad e_1^2 = e_1, \qquad e_2^2 = e_2.$$

Also $P_n^{(i)} = (\mathfrak{A}_P)e_i$. Since J permutes the factors $P_n^{(i)}$, $e_1^J = e_2$, $e_2^J = e_1$. It follows that J induces a nontrivial automorphism in $\Phi(q)$. Hence J is an involution of second kind. As before we see that $\mathfrak{K}$ is the set of J symmetric elements of $\mathfrak{A}$. We have therefore proved the following theorem.

THEOREM 4. *Let $\mathfrak{K}$ be a Jordan algebra over Φ such that there exists a finite extension P of Φ such that $\mathfrak{K}_P \cong P_{nj}$. Then either $\mathfrak{K}$ is isomorphic to a Jordan algebra $\mathfrak{A}_j$, $\mathfrak{A}$ central simple over Φ, or $\mathfrak{K}$ is isomorphic to the Jordan algebra of J-symmetric elements of a simple algebra $\mathfrak{A}$ that has center a quadratic extension $\Phi(q)$ of Φ and that possesses an involution J of second kind.*

In the first case we say that $\mathfrak{K}$ is of *type* A_I and in the second that $\mathfrak{K}$ is of *type* A_{II}.

Suppose now that $\mathfrak{K}$ is a Jordan algebra over Φ that has the following property: There exists an extension Γ of Φ such that $\mathfrak{K}$ can be regarded as an algebra over Γ with scalar multiplication an extension of the scalar multiplication over Φ and such that $\mathfrak{K}$ over Γ is of type A, B, C, or D. Now our results give the structure of $\mathfrak{K}$ regarded as an algebra over Γ. Thus if $\mathfrak{K}$ over Γ is of type A_I then we know that $\mathfrak{K}$ over Γ is isomorphic to an algebra $\mathfrak{A}_j$ where $\mathfrak{A}$ is central simple over Γ. But then $\mathfrak{K}$ (over Φ) is isomorphic to $\mathfrak{A}_j$ regarded as an algebra over Φ. Hence $\mathfrak{K}$ is isomorphic to ($\mathfrak{A}$ over $\Phi)_j$. If we regard $\mathfrak{A}$ from the beginning as an algebra over Φ we see that the center of $\mathfrak{A}$ is Γ and that $\mathfrak{K}$ is isomorphic to $\mathfrak{A}_j$. Next let $\mathfrak{K}$ over Γ be of type A_{II}. Then our discussion shows that we can find an associative algebra $\mathfrak{A}$ over Φ that has an involution J of second kind such that $\mathfrak{K}$ is isomorphic to the Jordan algebra $\mathfrak{H}(\mathfrak{A}, J)$ of J-symmetric elements of $\mathfrak{A}$. The center of $\mathfrak{A}$ is a quadratic extension $\Gamma(q)$ where Γ is the subfield of symmetric elements of the center. Similarly if $\mathfrak{K}$ over Γ is of type B or C then there is an involutorial simple associative algebra with center Γ and involution J of first kind such that $\mathfrak{K}$ is isomorphic to $\mathfrak{H}(\mathfrak{A}, J)$.

5. Universal algebras of Jordan algebras of types A–D.

To determine these algebras we shall make use of the following two lemmas.

LEMMA 1. *If R is an imbedding of a Jordan algebra $\mathfrak{K}$ and the enveloping algebra of R has the same dimensionality as the universal algebra then R is a universal imbedding.*

We know that the correspondence $a \rightarrow a^R$ can be extended to a homomorphism R of $\mathfrak{U}$ into the enveloping algebra $\mathfrak{E}$ of R. Since the dimensionalities of $\mathfrak{U}$ and $\mathfrak{E}$ are finite and equal, R is an isomorphism.

This lemma shows that a universal imbedding can be characterized as one that has maximum dimensionality for its enveloping algebra.

LEMMA 2. *Let R be an imbedding of $\mathfrak{K}$ in $\mathfrak{A}$ and let P be an extension of the base field Φ. Then R can be extended in one and only one way to an imbedding of $\mathfrak{K}_{\mathrm{P}}$ in $\mathfrak{A}_{\mathrm{P}}$. The dimensionality of the enveloping algebra of this extension is the same as that of R.*

If $x_1, x_2, \cdots$ is a basis for $\mathfrak{K}$ over Φ an imbedding is determined by associating with the x_i elements x_i^R that satisfy the same Jordan multiplication table as the x_i. For if the x_i^R are given with this property then it is clear that the correspondence $\sum \xi_i x_i \rightarrow \sum \xi_i x_i^R$ is an imbedding. Since the x_i also constitute a basis for $\mathfrak{K}_{\mathrm{P}}$ over P this proves the first assertion. Next let $z_1, z_2, \cdots$ be a basis for the enveloping algebra of R. It is immediate that every element of the enveloping algebra of the extension of R is a linear combination with coefficients in P of these elements. Also we know that since the z_i are Φ-independent then they are also P-independent. Hence the z's constitute a basis for the enveloping algebra of the extension.

We suppose now that $\mathfrak{A}$ is a simple associative algebra over Φ and that the center of $\mathfrak{A}$ is a field Γ that is separable over Φ. We wish to determine the universal algebra of the Jordan algebra $\mathfrak{A}_j$. Suppose first that $\mathfrak{A}=\Gamma$. Then $\Gamma=\Phi(\theta)$, θ a primitive element. Also since Γ is commutative the Jordan product in Γ coincides with ordinary multiplication. Hence $\Gamma_j=\Gamma$. Let R be a Jordan representation of Γ. Then $(\theta^k)^R=(\theta^{\cdot k})^R=(\theta^R)^{\cdot k}=(\theta^R)^k$. It follows that R is a homomorphism of the associative algebra Γ. Hence Γ is the universal algebra of Γ_j(15).

Assume next that $(\mathfrak{A}:\Gamma)>1$. Let $\mathfrak{A}'$ be an algebra anti-isomorphic to $\mathfrak{A}$ and let $a\rightarrow a'$ be a particular anti-isomorphism of $\mathfrak{A}$ onto $\mathfrak{A}'$. We form the direct sum $\mathfrak{B}=\mathfrak{A}\oplus\mathfrak{A}'$ and we let $\mathfrak{K}$ be the subset of $\mathfrak{B}$ of elements $a+a'$, $a\in\mathfrak{A}$. It is clear that $\mathfrak{K}$ is a subalgebra of $\mathfrak{B}_j$ isomorphic to $\mathfrak{A}_j$. We assert that $\mathfrak{K}$ generates $\mathfrak{B}$. For let a and b be two elements of $\mathfrak{A}$ such that $ab\neq ba$. Then

$$c = ab - ba = (ab + b'a') - (b + b')(a + a')$$

is in the algebra $\mathfrak{E}$ generated by $\mathfrak{K}$. The element $\sum a_i c b_i = \sum (a_i + a_i')c(b_i + b_i')$ is also contained in $\mathfrak{E}$. Since c is $\neq 0$ and $\mathfrak{A}$ is simple this means that every element of $\mathfrak{A}$ is in $\mathfrak{E}$. Similarly $\mathfrak{A}'\subseteq\mathfrak{E}$ and $\mathfrak{B}=\mathfrak{E}$.

We wish to show that the identity mapping is a universal imbedding of $\mathfrak{K}$ and hence that $\mathfrak{B}$ is a universal algebra. If $n^2=(\mathfrak{A}:\Gamma)>1$ and $r=(\Gamma:\Phi)$ then $(\mathfrak{A}:\Phi)=n^2r$ and $(\mathfrak{B}:\Phi)=2n^2r$. It therefore suffices to prove that if R is

(15) If Γ is a commutative algebra the universal algebra of Γ_j need not coincide with Γ. Thus let Γ have the basis x, y with $x^2=y^2=xy=yx=0$. Then the universal algebra has basis X, Y, $XY=-YX$.

any imbedding of $\mathfrak{A}_j$ then the dimensionality of the enveloping algebra is not greater than $2n^2r$.

Let Ω be the algebraic closure of Φ and form the algebra $\mathfrak{A}_\Omega$. Since Γ is separable over Φ, $\Gamma_\Omega = \Omega^{(1)} \oplus \Omega^{(2)} \oplus \cdots \oplus \Omega^{(r)}$ where the $\Omega^{(i)}$ are one-dimensional over Ω. It follows that $\mathfrak{A}_\Omega = \Omega_n^{(1)} \oplus \Omega_n^{(2)} \oplus \cdots \oplus \Omega_n^{(r)}$ where each $\Omega_n^{(i)}$ is a complete matrix algebra of n rows. Also $(\mathfrak{A}_j)_\Omega = (\mathfrak{A}_\Omega)_j = \Omega_{nj}^{(1)} \oplus \Omega_{nj}^{(2)} \oplus \cdots \oplus \Omega_{nj}^{(r)}$ for the Jordan algebras. By Theorem 2 and our discussion of the universal algebras of the algebras Ω_{nj} we see that the dimensionality of the universal algebra of $(\mathfrak{A}_j)_\Omega$ is $2n^2r$. By Lemma 2 the dimensionality of any enveloping algebra of $\mathfrak{A}_j$ does not exceed $2n^2r$. This completes the proof of the following theorem.

THEOREM 5. *Let $\mathfrak{A}$ be a simple associative algebra with center Γ separable over the base field Φ. Assume that $(\mathfrak{A}:\Gamma) > 1$. Let $\mathfrak{A}'$ be anti-isomorphic to $\mathfrak{A}$ and let $a \rightarrow a'$ be a particular anti-isomorphism of $\mathfrak{A}$ onto $\mathfrak{A}'$. Then if $a \rightarrow a^R$ is any imbedding of $\mathfrak{A}_j$, the mapping $a + a' \rightarrow a^R$ defined on a subset of $\mathfrak{B} = \mathfrak{A} \oplus \mathfrak{A}'$ can be extended to a homomorphism of $\mathfrak{B}$ into the enveloping algebra of R.*

We consider next an associative simple algebra $\mathfrak{A}$ over Φ that possesses an involution J. Let $\mathfrak{H}(\mathfrak{A}, J)$ be the totality of J-symmetric elements. Then $\mathfrak{H}(\mathfrak{A}, J)$ is a subalgebra of $\mathfrak{A}_j$. We again assume that the center Γ of $\mathfrak{A}$ is separable over Φ and we wish to determine the universal algebra of $\mathfrak{H}(\mathfrak{A}, J)$. Let Ω be the algebraic closure of Φ and consider the algebra $\mathfrak{A}_\Omega$. The involution J can be extended in one and only one way to an involution J in $\mathfrak{A}_\Omega$. The space of J-symmetric elements of $\mathfrak{A}_\Omega$ is the extension space $\mathfrak{H}(\mathfrak{A}, J)_\Omega$.

Any involution induces an automorphism in the center Γ and since $J^2 = 1$, the induced automorphism is either the identity or it has the period two in Γ. We recall that in the former case J is of first kind and in the second that J is of second kind. We suppose first that J is of second kind. We assume throughout that $(\mathfrak{A}:\Gamma) > 1$.

Let Δ be the subfield of Γ of J-symmetric elements and let $(\Delta:\Phi) = t$. Then $(\Gamma:\Delta) = 2$ and $(\Gamma:\Phi) = r = 2t$. Hence $\Gamma_\Omega = \Omega^{(1)} \oplus \Omega^{(2)} \oplus \cdots \oplus \Omega^{(2t)}$ where each $\Omega^{(i)}$ is one-dimensional. Since the $\Omega^{(i)}$ are uniquely determined, J permutes these components. Hence if e_i denotes the identity of $\Omega^{(i)}$ then e_i^J is one of the e's. Since J is of period two the permutation of the e's that it determines is of period two. Hence we may suppose that J has the form: $(e_1, e_2)(e_3, e_4) \cdots (e_{2s-1}, e_{2s})(e_{2s+1}) \cdots (e_{2t})$. Then a basis for the set of J-symmetric elements of Γ_Ω is $e_1 + e_2, \cdots, e_{2s-1} + e_{2s}, e_{2s+1}, \cdots, e_{2t}$. Hence the dimensionality of this set is $s + (2t - 2s) = 2t - s$. Since we know that the dimensionality of this set is the same as that of Δ, namely t, we see that $t = s$. It follows that J interchanges $\Omega_n^{(1)}$ and $\Omega_n^{(2)}$, $\Omega_n^{(3)}$ and $\Omega_n^{(4)}, \cdots$. The elements of $\mathfrak{H}(\mathfrak{A}, J)_\Omega$ are arbitrary sums of the form

$$(a_1 + \overset{J}{a_1}) + (a_3 + \overset{J}{a_3}) + \cdots + (a_{2t-1} + \overset{J}{a_{2t-1}}) \tag{56}$$

where $a_i \in \Omega_n^{(1)}$. This shows that $\mathfrak{H}(\mathfrak{A}, J)_\Omega \cong \Omega_{nj}^{(1)} \oplus \Omega_{nj}^{(3)} \oplus \cdots \oplus \Omega_{nj}^{(2t-1)}$. Using our results we know that the dimensionality of the universal algebra of such a direct sum is $2tn^2$. On the other hand it is easy to see by the argument used in the proof of Theorem 5 that the subalgebra generated by the elements of the form (56) is the whole algebra $\mathfrak{A}_\Omega$. Hence the subalgebra generated by $\mathfrak{H}(\mathfrak{A}, J)$ is the whole of $\mathfrak{A}$. Since $(\mathfrak{A}:\Phi) = 2tn^2$ we see as before that $\mathfrak{A}$ is the universal algebra of $\mathfrak{H}(\mathfrak{A}, J)$. It is also clear that J is the fundamental involution in the universal algebra.

We assume next that J is of first kind. Here $\mathfrak{H}(\mathfrak{A}, J)$ contains Γ and we can regard $\mathfrak{A}$ and $\mathfrak{H}$ as algebras over Γ. We do this first and we consider $(\mathfrak{A}$ over $\Gamma)_\Omega = \Omega_n$. The involution J can be extended to an involution J in Ω_n and the symmetric elements relative to the extension constitute the algebra $(\mathfrak{H}$ over $\Gamma)_\Omega$. On the other hand since Ω is algebraically closed we know that we can choose a suitable matrix basis for Ω_n so that J appears to have either the form $a \to a'$ or $a \to q^{-1}a'q$ (n even) where q is given by (4). Accordingly we say that J is of *type* B or of *type* C. We exclude the case type C and $n \leqq 4$ from further consideration. Using the values for the dimensionalities of the algebras of class B and C we obtain

$$(\mathfrak{H}:\Gamma) = ((\mathfrak{H} \text{ over } \Gamma)_\Omega:\Omega) \quad \begin{cases} n(n+1)/2 & \text{for type B} \\ n(n-1)/2 & \text{for type C.} \end{cases}$$

Hence $(\mathfrak{H}:\Phi) = rn(n+1)/2$ or $rn(n-1)/2$ in the respective cases if $r = (\Gamma:\Phi)$.

We now regard $\mathfrak{A}$, Γ, and $\mathfrak{H}$ as algebras over Φ. We have $\Gamma_\Omega = \Omega^{(1)} \oplus \Omega^{(2)} \oplus \cdots \oplus \Omega^{(r)}$, $\mathfrak{A}_\Omega = \Omega_n^{(1)} \oplus \Omega_n^{(2)} \oplus \cdots \oplus \Omega_n^{(r)}$. The involution J can be extended to an involution in $\mathfrak{A}_\Omega$. This mapping leaves each $\Omega^{(i)}$ and hence each $\Omega_n^{(i)}$ fixed. Hence we may suppose that in $\Omega_n^{(i)}$ we have either $a_i^J = a_i'$ or $a_i^J = q^{-1}a_i'q$, q as in (4). It is clear that $\mathfrak{H}(\mathfrak{A}, J)_\Omega$ is a direct sum of the algebras of J-symmetric elements of the $\Omega_n^{(i)}$. Since $(\mathfrak{H}(\mathfrak{A}, J)_\Omega:\Omega) = (\mathfrak{H}(\mathfrak{A}, J):\Phi) = rn(n+1)/2$ or $rn(n-1)/2$ it follows that we either have $a_i^J = a_i'$ for all i or $a_i^J = q^{-1}a_i'q$ for all i. In no case do we have $a_i^J = q^{-1}a_i'q$ if $n = 4$. Hence the dimensionality of the universal algebra of $\mathfrak{H}_\Omega$ is rn^2. Also we see that the subalgebra of $\mathfrak{A}_\Omega$ generated by $\mathfrak{H}_\Omega$ is $\mathfrak{A}_\Omega$. Hence the subalgebra generated by $\mathfrak{H}$ is $\mathfrak{A}$. Since $(\mathfrak{A}:\Phi)$ is rn^2 we see that $\mathfrak{A}$ is the universal algebra. It is also clear that J is the fundamental involution in the universal algebra. This completes the proof of the following theorem.

THEOREM 6. *Let $\mathfrak{A}$ be a simple associative algebra that has separable center Γ and that possesses an involution J. Assume that* $(\mathfrak{A}:\Gamma) > 1$ *and that* $(\mathfrak{A}:\Gamma) > 4$ *if J is of first kind and type* C. *Then if $\mathfrak{H}(\mathfrak{A}, J)$ denotes the subalgebra of $\mathfrak{A}_j$ of J-symmetric elements, any imbedding of $\mathfrak{H}(\mathfrak{A}, J)$ can be extended to a homomorphism of the associative algebra $\mathfrak{A}$.*

We consider finally the Jordan algebras of type D. Let $\mathfrak{A}$ be a Clifford algebra over a field Γ that is separable over Φ and let $\mathfrak{R}$ be the Jordan subalgebra of $\mathfrak{A}_j$ of Γ-combinations of the elements $s_0, s_1, \cdots, s_n$. We regard $\mathfrak{A}$ and $\mathfrak{K}$ as algebras over Φ. If $\xi_1, \xi_2, \cdots, \xi_r$ is a basis for Γ over Φ then the $(n+1)r$ elements $\xi_i s_j = \xi_i \cdot s_j$ form a basis for $\mathfrak{K}$ over Φ. Also we have the multiplication rule $(\xi_i \cdot s_j) \cdot (\xi_k \cdot s_i) = (\xi_i \cdot \xi_k) \cdot (s_j \cdot s_i)$. If Ω is the algebraic closure of Φ, $\Gamma_\Omega = \Omega^{(1)} \oplus \Omega^{(2)} \oplus \cdots \oplus \Omega^{(r)}$. Hence the space over Ω determined by the ξ's has a basis $e_1, e_2, \cdots, e_r$ with multiplication table $e_i \cdot e_j = \delta_{ij} e_i$. Hence $\mathfrak{K}_\Omega$ has the basis $e_i \cdot s_j$ such that $(e_i \cdot s_j) \cdot (e_k \cdot s_i) = \delta_{ik} e_i \cdot (s_j \cdot s_i)$. It follows that $\mathfrak{K}_\Omega = \mathfrak{K}^{(1)} \oplus \mathfrak{K}^{(2)} \oplus \cdots \oplus \mathfrak{K}^{(r)}$ where $\mathfrak{K}^{(i)}$ has the basis $e_i = e_i \cdot s_0$, $e_i \cdot s_1, \cdots$, $e_i \cdot s_n$. Any elements of Γ_Ω has the form $\sum \omega_i e_i$, ω_i in Ω. Hence the product of such an element by e_i is $\omega_i e_i$. It follows that if $j, k \geqq 1$ then

$$(e_i \cdot s_j)(e_i \cdot s_k) = e_i \cdot (s_j \cdot s_k) = \delta_{jk} \omega_j e_i.$$

Hence each $\mathfrak{K}^{(i)}$ is of type D. The dimensionality of the universal algebra of $\mathfrak{K}^{(i)}$ is 2^m; hence that of $\mathfrak{K}_\Omega$ is $r2^m$. On the other hand we see that the algebra $\mathfrak{A}$ is of dimensionality $r2^m$. This proves that $\mathfrak{A}$ is the universal algebra of $\mathfrak{K}$ over Φ.

THEOREM 7. *Let $\mathfrak{A}$ be an algebra of Clifford numbers over a field Γ that is separable over Φ and let $\mathfrak{K}$ be the Jordan subalgebra of $\mathfrak{A}_j$ whose basis over Γ is the set of generators $s_0, s_1, \cdots, s_n$. Then any imbedding of $\mathfrak{K}$ over Φ can be extended to a homomorphism of $\mathfrak{A}$ over Φ.*

The fundamental involution J in the universal algebra sends

$$\sum_{\epsilon_i=0,1} \rho_{\epsilon_1 \ldots \epsilon_n} s_1^{\epsilon_1} \cdots s_n^{\epsilon_n} \to \sum_{\epsilon_i=0,1} \rho_{\epsilon_1 \ldots \epsilon_n} s_n^{\epsilon_n} \cdots s_1^{\epsilon_1}$$

where the ρ's are in Γ. We remark that if $n \geqq 4$ then $\mathfrak{A}$ contains J-symmetric elements that do not belong to $\mathfrak{K}$. For example $s_1 s_2 s_3 s_4$ is such an element.

6. **Isomorphisms and derivations.** Let G be an ismorphism of a Jordan algebra $\mathfrak{K}_1$ on a Jordan algebra $\mathfrak{K}_2$. If $\mathfrak{U}_i$ is the universal algebra of $\mathfrak{K}_i$ then G can be extended to an isomorphism of $\mathfrak{U}_1$ on $\mathfrak{U}_2$. Also if J_i is the fundamental involution in $\mathfrak{U}_i$ then if $a_1 \in \mathfrak{K}_1$, $a_1 J_1 G = a_1 G = a_1 G J_2$. Thus the two anti-isomorphisms $J_1 G$ and $G J_2$ coincide on $\mathfrak{K}_1$. Since $\mathfrak{K}_1$ generates $\mathfrak{U}_1$ it follows that $J_1 G = G J_2$. Hence $J_2 = G^{-1} J_1 G$. In general if $\mathfrak{U}_i$ are associative algebras with involutions J_i then we say that J_1 and J_2 are *cogredient* if there exists an isomorphism G of $\mathfrak{U}_1$ on $\mathfrak{U}_2$ such that $J_2 = G^{-1} J_1 G$. Then we see that if $\mathfrak{K}_1$ and $\mathfrak{K}_2$ are isomorphic Jordan algebras then the fundamental involutions in their universal algebras are cogredient. Conversely suppose that the J_i are cogredient and, moreover, that $\mathfrak{K}_i$ is the complete set of J_i-symmetric elements of $\mathfrak{U}_i$. Then if $J_2 = G^{-1} J_1 G$, G maps $\mathfrak{K}_1$ on $\mathfrak{K}_2$. Hence G induces an isomorphism of $\mathfrak{K}_1$ on $\mathfrak{K}_2$. These remarks can be used to determine the isomorphisms between the Jordan algebras considered in the preceding section.

As before we shall assume that $(\mathfrak{A}:\Gamma) > 1$ for the involutorial algebras $\mathfrak{A}$ that define Jordan algebras of types $\mathrm{A_{II}}$, B and C. We assume also that

$(\mathfrak{A}:\Gamma)\geqq 6$ if the involution is of type C. Finally we restrict the value of n in the definition of the algebras of type D to $n\geqq 5$. Under these restrictions we can show that no algebra of one type can be isomorphic to one of a different type. We assume first that one of our algebras, say $\mathfrak{K}_1$, is of type D. If G is an isomorphism of $\mathfrak{K}_1$ on a second algebra in our list then $J_2=G^{-1}J_1G$ for the fundamental involutions. Hence G maps the totality $\mathfrak{L}_1$ of J_1-symmetric elements on the totality $\mathfrak{L}_2$ of J_2-symmetric elements. We know that $\mathfrak{L}_1\supset\mathfrak{K}_1$. Hence $\mathfrak{L}_2\supset\mathfrak{K}_2$. On the other hand we know that if $\mathfrak{K}_2$ has type A, B, or C then $\mathfrak{L}_2=\mathfrak{K}_2$. Thus $\mathfrak{K}_2$ is also of type D.

We note next that an algebra of type A_{I} can not be isomorphic to one of type A_{II}, B or C. For the universal algebras for types A_{I} are not simple while those of the other types are. An algebra of type A_{II} can not be isomorphic to one of type B or C since an involution of second kind cannot be cogredient to one of first kind. Finally let $\mathfrak{K}_1$ of type B be isomorphic to $\mathfrak{K}_2$ of type C. Then the universal algebras $\mathfrak{A}_i$ are isomorphic. If Γ_i is the center of $\mathfrak{A}_i$, $(\mathfrak{A}_1:\Gamma_1)=n^2=(\mathfrak{A}_2:\Gamma_2)$ and $(\Gamma_1:\Phi)=r=(\Gamma_2:\Phi)$. Then $(\mathfrak{K}_1:\Phi)=rn(n+1)/2$ while $(\mathfrak{K}_2:\Phi)=rn(n-1)/2$. This is impossible. Hence we have proved:

THEOREM 8. *Under the restrictions in the orders noted above Jordan algebras of different types are not isomorphic.*

Now let $a_1\rightarrow a_1^G$ be an isomorphism of $\mathfrak{A}_{1j}$ in $\mathfrak{A}_{2j}$ where $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are simple with separable centers. Then we form the direct sum of $\mathfrak{A}_1$ and $\mathfrak{A}_1'$ where $\mathfrak{A}_1'$ is anti-isomorphic to $\mathfrak{A}_1$ under the correspondence $a_1\rightarrow a_1'$. We know that $a_1+a_1'\rightarrow a_1^G$ defines a homomorphism G^* of $\mathfrak{B}_1=\mathfrak{A}_1\oplus\mathfrak{A}_1'$ on $\mathfrak{A}_2$. Since the only two-sided ideals in $\mathfrak{B}_1$ are $\mathfrak{B}_1$, $\mathfrak{A}_1$, $\mathfrak{A}_1'$ and 0, G^* either maps $\mathfrak{A}_1'$ on 0 or it maps $\mathfrak{A}_1$ on 0. In the first case $(a_1+a_1')^{G^*}=a_1^{G^*}=a_1^G$ so that G^* induces the original mapping of $\mathfrak{A}_1$ on $\mathfrak{A}_2$. Thus G is an isomorphism of the associative algebra $\mathfrak{A}_1$ on the associative algebra $\mathfrak{A}_2$. In the second case $(a_1+a_1')^{G^*}=(a_1')^{G^*}=a_1^G$. Hence G is the resultant of $a_1\rightarrow a_1'$ and G^*. Thus G is an anti-isomorphism of $\mathfrak{A}_1$ on $\mathfrak{A}_2$. This gives the following result which in a more general form is due to G. Ancochea([16]):

THEOREM 9. *Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be simple associative algebras with separable centers. Suppose that G is an isomorphism of $\mathfrak{A}_{1j}$ on $\mathfrak{A}_{2j}$. Then G is either an isomorphism or an anti-isomorphism of $\mathfrak{A}_1$ on $\mathfrak{A}_2$.*

In a similar manner we can easily prove the following theorem.

THEOREM 10. *Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be involutorial simple algebras with separable centers. Suppose that G is an isomorphism of $\mathfrak{H}(\mathfrak{A}_1, J_1)$ on $\mathfrak{H}(\mathfrak{A}_2, J_2)$, J_i the given involutions. Then G can be extended to an isomorphism G of $\mathfrak{A}_1$ on $\mathfrak{A}_2$ such that $J_2=G^{-1}J_1G$*([17]).

([16]) Ancochea [1]. See also N. Jacobson [4] and Kaplansky [1].

([17]) For central algebras this has been proved by Kalisch [1]. A more general result is given in N. Jacobson [4].

In the case of algebras of type D we restrict ourselves to the case $\Gamma=\Phi$. An isomorphism G of the system $(s_0, s_1, \cdots, s_n)$ onto $(t_0, t_1, \cdots, t_n)$ sends s_i into $s_i^G=\sum\mu_{ij}t_j$. Clearly $s_0^G=t_0$. It is easy to see that the only elements of $(s_0, s_1, \cdots, s_n)$ that satisfy quadratic equations of the form $x^2=\xi s_0$ are the multiples of s_0 and the elements of the form $\sum_1^n\xi_is_i$. It follows that $s_i^G=\sum_1^n\mu_{ij}t_j$ for $i=1, 2, \cdots, n$. Now if $x=\sum_1^n\xi_is_i$

$$x^2 = \sum_1^n \alpha_i\xi_i^2 \equiv f(\xi)$$

and $(x^G)^2=f(\xi)$. On the other hand $x^G=\sum\xi_is_i^G=\sum\xi_i\mu_{ij}t_j\equiv\sum\eta_jt_j$ so that $(x^G)^2=\sum_1^n\beta_j\eta_j^2$ if $t_j^2=\beta_js_0$. Thus the diagonal matrices $\alpha=\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ and $\beta=\{\beta_1, \beta_2, \cdots, \beta_n\}$ are cogredient. It is easily seen that this condition is also sufficient. Also one sees that the group of automorphisms of $(s_0, s_1, \cdots, s_n)$ is isomorphic to the group of matrices (μ) such that $(\mu)'\alpha(\mu)=\alpha$.

We shall obtain next the derivations of our algebras. These can be obtained by using the following general principle: If D is a derivation in a Jordan algebra $\mathfrak{R}$ then D can be extended to a derivation in its universal algebra $\mathfrak{U}$. As in §1 we take $\mathfrak{U}$ to be the algebra $\mathfrak{F}/\mathfrak{B}$ where $\mathfrak{F}$ is the free algebra generated by $X_1, X_2, \cdots$ in 1–1 correspondence with the basis $x_1, x_2, \cdots$ of $\mathfrak{R}$ and $\mathfrak{B}$ is the ideal generated by

$$(X_iX_j + X_jX_i)/2 - \sum \gamma_{ijk}X_k. \tag{58}$$

The element x_i is identified with the coset mod $\mathfrak{B}$ of X_i. If D is a derivation in $\mathfrak{R}$ we have $x_i^D=\sum\sigma_{ij}x_j$ where

$$(x_i^D x_j + x_ix_j^D + x_j^D x_i + x_jx_i^D)/2 = \sum \gamma_{ijk}x_k^D. \tag{59}$$

Now it is clear that since $\mathfrak{F}$ is a free algebra there exists a derivation sending the generators X_i into arbitrary Y_i in $\mathfrak{F}$. We take $Y_i=\sum\sigma_{ij}X_j$ and call D^* the associated derivation in $\mathfrak{U}$. Then D^* maps (58) into

$$(X_i^{D^*}X_j + X_iX_j^{D^*} + X_j^{D^*}X_i + X_jX_i^{D^*})/2 - \sum \gamma_{ijk}X_k^{D^*}.$$

The coset of this element mod $\mathfrak{B}$ is 0 by (59). Hence D^* maps $\mathfrak{B}$ into itself. Therefore D^* induces a derivation D in $\mathfrak{U}=\mathfrak{F}/\mathfrak{B}$. Clearly the induced mapping is an extension of the original D in $\mathfrak{R}$.

We apply this result first to the case $\mathfrak{R}=\mathfrak{A}_j$, $\mathfrak{A}$ simple with separable center. Then we see that D defines a derivation in $\mathfrak{B}=\mathfrak{A}\oplus\mathfrak{A}'$ that maps $a+a'$ into $a^D+(a^D)'$, a in $\mathfrak{A}$. Since any derivation in $\mathfrak{B}$ is inner[18] there exists an element $d+e'$, $d\in\mathfrak{A}$, $e'\in\mathfrak{A}'$ such that

$$a^D + (a^D)' = [a + a', d + c'] = [a, d] + [a', c']$$

[18] N. Jacobson [1, p. 215].

where as usual $[x, y]$ denotes $xy-yx$. It follows that $a^D=[a, d]$. This proves the following theorem.

THEOREM 11. *Let* $\mathfrak{A}$ *be a simple associative algebra with separable center. Then if D is a derivation in the Jordan algebra* $\mathfrak{A}_j$ *there exists an element d in* $\mathfrak{A}$ *such that* $a^D=[a, d]$ *for all a.*

Suppose next that $\mathfrak{A}$ has an involution and that D is a derivation in the Jordan algebra $\mathfrak{H}(\mathfrak{A}, J)$. Then D defines a derivation in the universal algebra $\mathfrak{A}$. Hence there exists a $d\in\mathfrak{A}$ such that $a^D=[a, d]$ for all $a\in\mathfrak{H}$. We apply J to this equation and obtain $a^D=[d^J, a]=[-a, d^J]$. Hence $d+d^J$ commutes with every $a\in\mathfrak{H}$. It follows that $d+d^J=\delta\in\Gamma$. Since $\delta^J=\delta$ we can replace d by $d'=d-\delta/2$ to obtain $(d')^J=-d'$. This element produces the same effect as d; hence we can suppose that $d^J=-d$.

THEOREM 12. *Let* $\mathfrak{A}$ *be a simple associative algebra that has a separable center and that has an involution J. Then if D is a derivation in* $\mathfrak{H}(\mathfrak{A}, J)$ *there exists a J-skew element d in* $\mathfrak{A}$ *such that* $a^D=[a, d]$.

In considering the derivations of the algebras of type D we again restrict ourselves to the central simple case $\Gamma=\Phi$. Actually it can be seen that this restriction is not necessary but for the sake of brevity we make it here. Let D be a derivation in the system $(s_0, s_1, \cdots, s_n)$. Since $s_0^2=s_0$, $2s_0\cdot s_0^D=s_0^D$. Hence $s_0^D=0$. Next set $x=\sum_1^n\xi_i s_i$ and differentiate $x^2=f(\xi)$. This gives $x\cdot(x^D)=0$. If we apply this to $x=s_i$ we see that $s_i^D=\sum_{j=1}^n\sigma_{ij}s_j$. Then $x\cdot(x^D)=0$ implies that the matrix (σ) satisfies the equation $\alpha(\sigma)+(\sigma)'\alpha=0$, $\alpha=\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ as before.

7. **Semi-simple Jordan algebras of characteristic** 0. We recall that Albert[19] has called a Jordan algebra *solvable* if the *derived* sequence $\mathfrak{R}$, $\mathfrak{R}^{\cdot 2}$, $\mathfrak{R}^{\cdot 4}=(\mathfrak{R}^{\cdot 2})^{\cdot 2}, \cdots$ leads to 0. Here $\mathfrak{R}^{\cdot 2}$ is the space generated by all products $a\cdot b$, a and b in $\mathfrak{R}$. The *radical* of a Jordan algebra $\mathfrak{R}$ is the maximal solvable ideal and $\mathfrak{R}$ is *semi-simple* if it has no nonzero solvable ideals. It has been proved by Albert that any semi-simple Jordan algebra over a field of characteristic 0 has an identity and is a direct sum of simple algebras. This result reduces the problem of determining the semi-simple algebras to that of determining the simple algebras.

Now let $\mathfrak{R}$ be simple. Let Γ be the center of $\mathfrak{R}$ defined in the usual manner for non-associative rings as the totality of elements that commute and associate with all the elements of the algebra. Then Γ is the set of elements such that

$$\gamma\cdot(a\cdot b)=(\gamma\cdot a)\cdot b,\qquad a\cdot(\gamma\cdot b)=(a\cdot\gamma)\cdot b,$$
$$a\cdot(b\cdot\gamma)=(a\cdot b)\cdot\gamma\text{[20]}.$$

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(19) Albert's results quoted here are in [4] and [2].

(20) Cf. Albert [5] or N. Jacobson [3].

It is known that Γ is a field containing the set $\Phi 1$ of Φ-multiples of 1. By (60) $\mathfrak{K}$ can be regarded as an algebra over Γ with scalar multiplication an extension of the originally defined multiplication $\alpha a = \alpha 1 \cdot a$. It is known that $\mathfrak{K}$ is central simple over Γ in the sense that ($\mathfrak{K}$ over $\Gamma)_\Omega$ is simple for every extension field Ω of Γ.

Now it has been shown by Albert [21] that there exists a field P such that ($\mathfrak{K}$ over $\Gamma)_P$ is a split algebra. Hence $\mathfrak{K}$ over Γ is of type A, B, C, D or E in our sense. If $\mathfrak{K}$ is of type E it is not a special Jordan algebra. On the other hand if $\mathfrak{K}$ has type A, B, C, or D then the results of §4 show that $\mathfrak{K}$ is a special Jordan algebra and we have the following structure theorem:

THEOREM 13. *Let $\mathfrak{K}$ be a simple special Jordan algebra over a field Φ of characteristic* 0. *Then $\mathfrak{K}$ is isomorphic to one of the following types of algebras*: (1) *an algebra $\mathfrak{A}_j$ where $\mathfrak{A}$ is a simple associative algebra*, (2) $\mathfrak{H}(\mathfrak{A}, J)$ *the algebra of J-symmetric elements of a simple associative algebra that possesses an involution J*, (3) *a Jordan algebra associated with a Clifford system over a field Γ containing Φ*.

In each of these cases we have proved that the universal algebra is either a simple algebra or it is a direct sum of two anti-isomorphic simple algebras. If $\mathfrak{K}$ is a semi-simple special Jordan algebra, $\mathfrak{K}$ is a direct sum of simple Jordan algebras. Also since any subalgebra of a special Jordan algebra is special, the simple components are special. Since the universal algebra of a direct sum of algebras with identities is a direct sum of the universal algebras of the components, we can state the following general theorem:

THEOREM 14. *The universal algebra of any semi-simple special Jordan algebra over a field of characteristic* 0 *is a semi-simple associative algebra.*

As a corollary we have the following result:

THEOREM 15. *Any matrix representation of a semi-simple Jordan algebra over a field of characteristic* 0 *is completely reducible.*

For it is known that any matrix representation of a semi-simple associative algebra is completely reducible.

Finally we note that we can use our determination of the universal algebras to obtain the irreducible matrix representations.

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(21) Albert [4, p. 567] and [2, p. 554].

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YALE UNIVERSITY,
NEW HAVEN, CONN.